Math 978 HW #1 Solutions.

1. S^3 is connected.

$\boxed{\text{Pf \#1:}}$ $S^3 = \mathbb{R}^3 \cup \{\infty\}$ = the 1-point compactification of $\mathbb{R}^3$, so $\mathbb{R}^3 \subseteq S^3$ and $\overline{\mathbb{R}^3} = S^3$. But $\mathbb{R}^3$ is (path-) connected; a path from $\vec{x}$ to $\vec{y}$ is given by $\gamma(t) = (1-t)\vec{x} + t\vec{y}$, $0 \le t \le 1$. So since $\mathbb{R}^3 \subseteq S^3 \subseteq \overline{\mathbb{R}^3}$ and $\mathbb{R}^3$ is connected, S^3 is connected by a result from Math 970.

$\boxed{\text{Pf \#2}}$ S^3 = unit sphere in $\mathbb{R}^4$. But $\mathbb{R}^4 \setminus \underline{0}$ is (path-) connected; a path from $\vec{x} \ne \underline{0}$ to $\vec{y} \ne \underline{0}$ is given by $\gamma(t) = (1-t)\vec{x} + t\vec{y}$, unless $\underline{0}$ lies on this line segment (i.e., $\frac{x}{\|x\|} = \frac{-y}{\|y\|}$). In that case, the path from $\vec{x} = (x,y,z,w)$ to $\vec{z} = (-y,x,-w,z)$ (which is $\perp$ to $\vec{x}$), followed by the path from $\vec{z}$ (which is $\perp$ to $\vec{y}$!) to $\vec{y}$ gives a path missing $\underline{0}$.

Then the map $f: \mathbb{R}^4 \setminus \underline{0} \to S^3$ given by $f(\vec{x}) = \frac{\vec{x}}{\|\vec{x}\|}$ is continuous (because $\vec{x}$ and $\|\vec{x}\|$ are cts, and $\|\vec{x}\|$ is never 0), with image S^3. So S^3 is the cts image of a connected space, so S^3 is connected.

2. If $x,y \in S^3$ then $\exists$ homeo $h: S^3 \to S^3$ with $h(x)=y$.

$\boxed{Pf \#1}$: The relation $x \sim y$ if $\exists$ homeo $h: S^3 \to S^3$ with $h(x)=y$ is an equivalence relation:

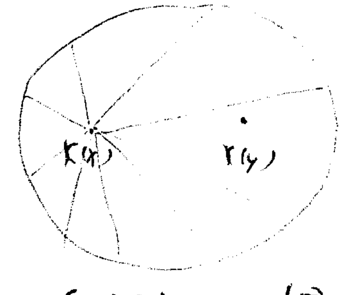
$x \sim x$ because $h = Id_{S^3}$ works

$x \sim y \Rightarrow y \sim x$ because $h(x)=y \Rightarrow h^{-1}(y)=x$ and $h^{-1}: S^3 \to S^3$ is a homeo

$x \sim y, y \sim z \Rightarrow x \sim z$ because $h,k: S^3 \to S^3$ homeos with $h(x)=y$, $k(y)=z \Rightarrow (k \circ h)(x) = k(h(x)) = k(y) = z$ and $k \circ h$ is a homeo.

S^3 is therefore partitioned by $\sim$ into disjoint equivalence classes.

But! Each equiv. class is an open subset of S^3: if V is an equiv class, $x \in V$ and $\mathcal{U}$ = a nbhd of x in S^3 homeomorphic to $\mathbb{R}^3$, then we can define a homeo map, for any $y \in \mathcal{U}$, $h_y: S^3 \to S^3$ with $h(x)=y$ as follows: for $k: \mathcal{U} \xrightarrow{\cong} \mathbb{R}^3$ draw a large ball $\mathbb{B}$ containing $k(x), k(y)$.



Any point $z \in \mathbb{B}$ can be expressed uniquely as $(1-t)k(x) + t\vec{v}$ for $t \in [0,1]$, $\vec{v} \in \partial\mathbb{B}$.

Then define $l: \mathbb{B} \to \mathbb{B}$ by $l((1-t)k(x) + t\vec{v}) = (1-t)k(y) + t\vec{v}$

This fcn is cts, a small change in $(1-t)k(x) + t\vec{v}$ means a small change in t and $\vec{v}$ means a small change in $(1-t)k(y) + t\vec{v}$.

Note that $\ell|\partial\mathbb{B} = Id_{\partial\mathbb{B}}$. Now define a map

$h_y : S^3 \to S^3$ by $h_y(z) = \begin{cases} K^{-1}\ell K(z) & \text{if } z \in K^{-1}(\mathbb{B}) \\ z & \text{otherwise} \leftarrow z \in \overline{S^3 \setminus K^{-1}(\mathbb{B})} \end{cases}$

Since $\mathbb{B}$ is compact, $K^{-1}(\mathbb{B})$ is compact so $K^{-1}(\mathbb{B}) \subseteq S^3$ is closed. By the pasting lemma, h_y is cts. It is also a bijection from S^3 (compact) to S^3 (Hausdorff) so it is a homeo.

Finally, $h_y(x) = K^{-1}\ell K(x) = K^{-1}\ell(1 \cdot K(x) + 0 \cdot \vec{v}) = K^{-1}(1 \cdot K(y) + 0\vec{v})$
$= K^{-1}(K(y)) = y$.

So $U \subseteq V$ so every $x \in V$ has an open nbhd $\subseteq V$, so V is open

So every equivalence class is open, so S^3 has been expressed as a disjoint union of open sets; since S^3 is connected, all but one of those sets is empty. So $\sim$ has only one equivalence class, so $\forall\, x, y \in S^3$, $x \sim y$, i.e. $\exists$ homeo $h : S^3 \to S^3$ with $h(x) = y$.

$\boxed{\text{Pf \#2:}}$ $S^3 =$ unit sphere in $\mathbb{R}^4$. Given any $\vec{x} \in S^3$ we can extend $\vec{x}$ to a basis for $\mathbb{R}^4$ (if $\vec{x} = (x, y, z, w)$ then WLOG $x \neq 0$, so $(0,1,0,0), (0,0,1,0), (0,0,0,1)$ extend $\vec{x}$ to a basis). Then we may apply Gram-Schmidt (using $\bar{x}$ as first vector to build an orthonormal basis extending $\vec{x}$ ($\bar{x}$ is unchanged because $\|\bar{x}\| = 1$). Let $\vec{x}, \vec{y}, \vec{z}, \vec{w}$ be this o.n. basis. Then the matrix $M = (\vec{x}\ \vec{y}\ \vec{z}\ \vec{w})$ is orthogonal ($M^TM = Id$), so it takes unit vectors to unit vectors, i.e. gives a cts, bijective map
↖ it's linear ↖ it's invertible
$h : S^3 \to S^3$ with $h(1,0,0,0) = \vec{x}$. So in the equiv relation above, all $\vec{x} \sim (1,0,0,0)$, so there is only one equivalence class. ///

3. Any homeo $h: \mathbb{R}^3 \to \mathbb{R}^3$ extends to a homeo $\bar{h}: S^3 \to S^3$.

Pf #1 Quote Munkres, section 29, problem #5.

Pf #2 Prove M., §29, #5!

Given $h: \mathbb{R}^3 \to \mathbb{R}^3$ a homeo, define

$\bar{h}: S^3 = \mathbb{R}^3 \cup \{\infty\} \longrightarrow \mathbb{R}^3 \cup \{\infty\} = S^3$ by

$$\bar{h}(x) = \begin{cases} h(x) & \text{if } x \in \mathbb{R}^3 \\ \infty & \text{if } x = \infty \end{cases}$$

$\bar{h}$ is cts: If $V \subseteq S^3$ is open then either

$V \subseteq \mathbb{R}^3$ is open in $\mathbb{R}^3$, so $\bar{h}^{-1}(V) = h^{-1}(V)$ is open in $\mathbb{R}^3$ (because $\infty \notin V$

and $\infty \notin h^{-1}(V)$, so $h^{-1}(V)$ is open in S^3, or

$\infty \in V$ and $\mathbb{R}^3 \setminus V$ is compact, so

$h^{-1}(\mathbb{R}^3 \setminus V)$ is compact and $\infty \in \bar{h}^{-1}(V)$, so

$\mathbb{R}^3 \setminus \bar{h}^{-1}(V) = \mathbb{R}^3 \setminus h^{-1}(V) = h^{-1}(\mathbb{R}^3 \setminus V)$ is compact,

so $\bar{h}^{-1}(V)$ is open in S^3. So $V \subseteq S^3$ open $\Rightarrow \bar{h}^{-1}(V) \subseteq S^3$ open,

so $\bar{h}$ is a cts bijection (because h is) from S^3 cpct

to S^3 Hausdorff, so $\bar{h}$ is a homeo.

Finally $\bar{h}|_{\mathbb{R}^3} = h$ (by definition). ///

4. $K \subseteq S^3$ knot $x, y \in S^3$ $K \cap \{x,y\} = \phi$. Then $S^3 \setminus x, S^3 \setminus y \cong \mathbb{R}^3$ and K, thought of as $K_1, K_2 \subseteq$ these two $\mathbb{R}^3$'s, has $K_1 \underset{u}{=} K_2$.

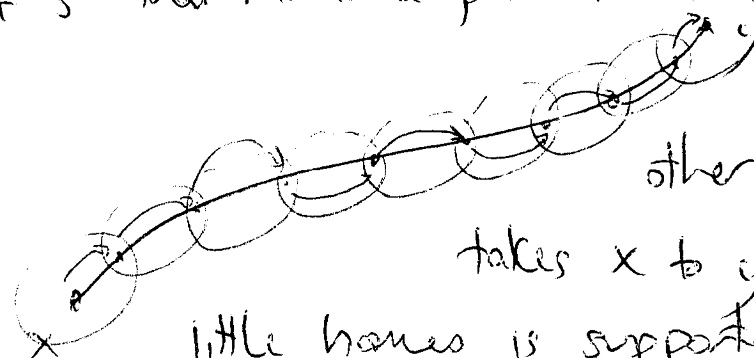
Pf: We have a homeo $h_x : S^3 \to S^3$ with $h(x) = \infty$, which restricts to a homeo $h_x : S^3 \setminus x \to S^3 \setminus \infty = \mathbb{R}^3$.

Similarly, we have $h_y : S^3 \setminus y \to \mathbb{R}^3$ a homeo.

Set $K_1 = h_x(K) \subseteq \mathbb{R}^3$, $K_2 = h_y(K) \subseteq \mathbb{R}^3$. We want a homeo $h : \mathbb{R}^3 \to \mathbb{R}^3$ with $h(K_1) = K_2$.

The natural choice is to take a homeo $h : S^3 \to S^3$ with $h(x) = y$, which restricts to $h : S^3 \setminus x \to S^3 \setminus y$ a homeo, and then use $\mathbb{R}^3 \xrightarrow{h_x^{-1}} S^3 \setminus x \xrightarrow{h} S^3 \setminus y \xrightarrow{h_y} \mathbb{R}^3$, which is what the instructor intended. But under this homeo,

$K_1 \xrightarrow{h_x^{-1}} K \xrightarrow{h} h(K) \xrightarrow{h_y} ???$! This carries K_1 to K_2 only if h carries K to K (then $h_y(K) = K_2$). The two proofs of Problem #2 are not sufficient to guarantee this! But the first proof method can be modified to do this, Because $S^3 \setminus K$ is (path-) connected. By choosing a path from x to y missing K, and covering it with little balls disjoint from K, and applying the construction given in Problem #2 to each, we can, by composing homeos of S^3 that moves a point from one end of the path



in each ball to the other, build a homeo that takes x to y which, since each little homeo is supported (= moves pts) only ~~pts~~ in its little ball, is supported on the union of these balls, which is disjoint from K. I.e. $h|_K = Id$, so $h(K)=K$. Using this homeo, the above argument works, and $h_y \circ h \circ h_x^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ carries K_1 to K_2. ///

5. $K_1, K_2 \subseteq \mathbb{R}^3 \subseteq S^3 = \mathbb{R}^3 \cup \{\infty\}$, then $\exists$ homeo $h : \mathbb{R}^3 \to \mathbb{R}^3$ with $h(K_1) = K_2 \iff \exists$ homeo $\bar{h} : S^3 \to S^3$ with $\bar{h}(K_1) = K_2$.

Pf: ($\Longrightarrow$) is problem #3. $h : \mathbb{R}^3 \to \mathbb{R}^3$ extends to a homeo $\bar{h} : S^3 \to S^3$, since $h(K_1) = K_2$, $\bar{h}(K_1) = h(K_1) = K_2$.

($\Longleftarrow$) Given $\bar{h} : S^3 \to S^3$ with $h(K_1) = K_2$, ~~we m~~ it might not come from a homeo $h : \mathbb{R}^3 \to \mathbb{R}^3$, i.e. $\bar{h}(\infty)$ might not be ∞. If $\bar{h}(\infty) = \infty$, then take the restriction, to give h. If $\bar{h}(\infty) = x \neq \infty$, then we have two points in S^3 to worry about, and so we use problem #4.

$\bar h$ gives a homeo $\bar h : S^3 \setminus \infty \longrightarrow S^3 \setminus x$ with $\bar h(K_1) = K_2$;

we then view K_2 as lying in both $\mathbb{R}^3 = S^3 \setminus \infty$ and $S^3 \setminus x$.

By Problem #4, if we write $S^3 \setminus \infty \cong \mathbb{R}^3$ via l_1, $l_1(K_2) = K_2'$

$S^3 \setminus x \cong \mathbb{R}^3$ via l_2, $l_2(K_2) = K_2''$, then $\exists$ homeo

$j : \mathbb{R}^3 \to \mathbb{R}^3$ with $j(K_2') = K_2''$

$$\begin{tikzcd}
\mathbb{R}^3 = S^3 \setminus \infty \arrow[r, "\bar h"] & S^3 \setminus x \arrow[r, "l_2"] & \mathbb{R}^3 \\
\cup & \cup & \cup \\
K_1 \arrow[r] & K_2 & K_2'' \\
& S^3 \setminus \infty \arrow[r, "l_1"] & \mathbb{R}^3 \arrow[uuu, bend right, "j"'] \\
& \cup & \cup \\
& K_2 & K_2'
\end{tikzcd}$$

Then

$$\begin{tikzcd}
\mathbb{R}^3 = S^3 \setminus \infty \arrow[r, "\bar h"] & S^3 \setminus x \arrow[r, "l_2"] & \mathbb{R}^3 \arrow[r, "j^{-1}"] & \mathbb{R}^3 \arrow[r, "l_1^{-1}"] & S^3 \setminus \infty = \mathbb{R}^3 \\
\cup & \cup & \cup & \cup & \cup \\
K_1 \arrow[r] & K_2 \arrow[r] & K_2'' \arrow[r] & K_2' \arrow[r] & K_2
\end{tikzcd}$$

$h = l_1^{-1} j^{-1} l_2 \bar h : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeo with $h(K_1) = K_2$. ∎