NOTES ON ALGEBRA

GRAPHS, TREES AND FREE GROUPS

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Definition 0.1. A *graph* Γ consists of

- a set $V = V(\Gamma)$ of *vertices*;
- a set $E = E(\Gamma)$ of *directed edges*;
- two functions $\alpha : E \to V$ and $\omega : E \to V$;
- a free involution of E, denoted $e \leftrightarrow \bar{e}$ such that $\alpha(e) = \omega(\bar{e})$.

An *(undirected) edge* is a pair $\{e, \bar{e}\}$.

Definition 0.2. An *edge-path* of length 0 in a graph Γ is a vertex. An edge-path of length $n > 0$ is a finite sequence $\gamma = (e_1, \ldots, e_n)$ of directed edges of Γ such that $\omega(e_i) = \alpha(e_{i+1})$ for $i = 1, \ldots, n-1$. The length of an edge-path γ will be denoted by $|\gamma|$.

If γ is an edge-path of length 0, consisting of the vertex v, then we set $\alpha(\gamma) = \omega(\gamma) = v$. If $\gamma = (e_1, \ldots, e_n)$ is an edge-path of positive length we wet $\alpha(\gamma) = \alpha(e_1)$ and $\omega(\gamma) = \omega(e_n)$. When $\alpha(\gamma) = v_1$ and $\omega(\gamma) = v_2$ then we say that γ *joins* v_1 to v_2.

If $\gamma = (e_1, \ldots, e_n)$ is an edge-path which joins v to w then $(\bar{e}_n, \ldots, \bar{e}_1)$ is an edge-path which joins w to v, and it will be denoted $\bar{\gamma}$. IF $|\gamma| = 0$ then $\bar{\gamma} = \gamma$.

If $\gamma = (e_1, \ldots, e_n)$ and $\delta = (f_1, \ldots, f_m)$ are edge-paths such that $\omega(\gamma) = \alpha(\delta)$, then the composite edge-path $(e_1, \ldots, e_n, f_1, \ldots, f_m)$ will be denoted $\gamma \star \delta$. If $|\gamma| = 0$ then $\gamma \star \delta = \delta$ if $\omega(\gamma) = \alpha(\delta)$, and $\delta \star \gamma = \delta$ if $\omega(\delta) = \alpha(\gamma)$.

We say that γ is a *geodesic* if $|\gamma| = 0$ or if $\gamma = (e_1, \ldots e_n)$ where $e_i \neq \bar{e}_{i+1}$ for $i = 1, \ldots, n-1$.

We say that γ is a *circuit* if γ is a geodesic such that $\alpha(\gamma) = \omega(\gamma)$ and $e_1 \neq \bar{e}_n$.

Definition 0.3. A graph Γ is *connected* if any two distinct vertices of Γ are joined by some edge-path.

Definition 0.4. A graph is called a *forest* if it contains no circuits. A connected forest is called a *tree*.

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1. The tree of words

Let X be a set (of "letters"). Let $\bar{X}$ be a disjoint set which is in $1-1$ correspondence with X. This correspondence determines an involution of $X \sqcup \bar{X}$ which we indicate by $x \leftrightarrow \bar{x}$.

Definition 1.1. A *word* in the alphabet X is a finite sequence $w = (x_1, \ldots, x_n)$ where $x_i \in X \sqcup \bar{X}$ for $i = 1, \ldots, n$. The *length* of w is $|w| = n$, and the empty sequence is considered to be a word of length 0. If $n > 0$ we will write $w = x_1 x_2 \cdots x_n$, but the word of length 0 is denoted by 1. The set of all words in the alphabet X will be denoted $\mathcal{W}(X)$.

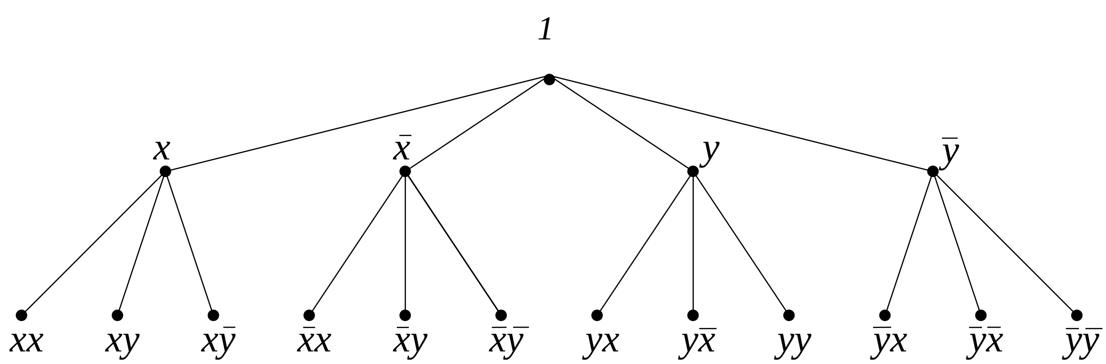
The word $w = x_1 x_2 \cdots x_n$ is said to be *reduced* if $x_i \neq \bar{x}_{i+1}$ for $i = 1, \ldots, n$. (And the word 1 is considered to be reduced.)

The concatenation operation on words will be denoted by $\diamond$. If $w = x_1 \cdots x_n$ and $v = y_1 \cdots y_m$ are two words in the alphabet X then define

$$w \diamond v = x_1 \cdots x_n y_1 \cdots y_m.$$

Naturally, we also define $1 \diamond w = w \diamond 1 = w$.

Next we construct a graph $T(X)$, which will turn out to be a tree, having as vertices the reduced words in the alphabet X. (The figure below shows the vertices up to length 2 in the case $X = \{x, y\}$.)



The set of directed edges of $T(X)$ is the set

$$E = \{(w, w \diamond x), (w \diamond x, w) \mid x \in X \sqcup \bar{X} \text{ and } w \diamond x \text{ is reduced } \}.$$

The functions α and ω send each pair to its first or second element respectively.

We will think of the directed edges of $T(X)$ as having *labels*: a directed edge of the form $(w, w \diamond x)$ has label x and a directed edge of the form $(w \diamond x, w)$ has label $\bar{x}$. In particular, if e has label x then $\bar{e}$ has label $\bar{x}$.

Proposition 1.2. *The graph $T(X)$ is a tree.*

Proof. It is clear that $T(X)$ is connected. If e is a directed edge of $T(X)$ then either $|\alpha(e)| < |\omega(e)|$ or $|\alpha(e)| > |\omega(e)|$. Let us say that e is *increasing* in the first case and *decreasing* in the second. (An edge never joins two vertices of the same length.) From the construction of $T(X)$ we see that if v is a vertex with $|v| > 1$ then there is exactly one increasing edge e with $\omega(e) = v$. In particular, this means that in a geodesic γ it is not possible for an increasing edge to be followed by a decreasing edge. But any circuit would have to contain an increasing edge followed by a decreasing edge, so there can be no circuit in $T(X)$. $\square$

By an *automorphism* g of a graph G we mean a bijection of $V(G) \sqcup E(G)$ that sends $V(G)$ to $V(G)$ and $E(G)$ to $E(G)$ such that $\alpha(g(e)) = g(\alpha(e))$ and $g(\bar{e}) = \overline{g(e)}$. The group of automorphisms of G will be denoted $\mathrm{Aut}(G)$.

For each $x \in X \sqcup \bar{X}$ there is an automorphism σ_x of $T(X)$ constructed by "moving the vertex $\bar{x}$ to the top". That is, for a reduced word w, define

$$\sigma_x(w) = \begin{cases} v & \text{if } w = \bar{x} \diamond v, \\ x \diamond w & \text{if } x \diamond w \text{ is reduced.} \end{cases}$$

Note that $\sigma_{\bar{x}}$ is the inverse of σ_x, regarded as a permutation of the vertices of $T(X)$.

To verify that σ_x extends (in a unique way) to an automorphism of $T(X)$ it suffices to prove the following;

Lemma 1.3. *Let $x \in X \sqcup \bar{X}$. The vertices v and w of $T(X)$ are joined by an edge with label y if and only if the vertices $\sigma_x(v)$ and $\sigma_x(w)$ are joined by an edge with label y.*

Proof. Suppose there is an edge e with label $y \in X \sqcup \bar{X}$ such that $\alpha(e) = v$ and $\omega(e) = w$. We have two cases: either $w = v \diamond y$ or $v = w \diamond \bar{y}$, where $y \in X$ is the label of e.

Suppose that $w = v \diamond y$. We have two subcases, according to whether $x \diamond v$ is reduced. If $x \diamond v$ is not reduced then $v = \bar{x} \diamond u$, and $\sigma_x(v) = u$. If $u \neq 1$ then $\sigma_x(w) = u \diamond y$ and we have $(\sigma_x(v), \sigma_x(w)) = (u, u \diamond y)$, which is an edge with label y. If $u = 1$ then $y \neq x$, since v is reduced, so we have $\sigma_x(w) = y$ and $(\sigma_x(v), \sigma_x(w)) = (1, y)$, which is an edge with label y. In the second subcase, where $x \diamond v$ is reduced, we have $\sigma_x(v) = x \diamond v$ and $\sigma_x(w) = x \diamond v \diamond y$, and $(\sigma_x(v), \sigma_x(w)) = (x \diamond v, x \diamond v \diamond y)$ is again an edge with label y.

The case $v = w \diamond \bar{y}$ is similar. $\square$

Definition 1.4. The *free group* $F(X)$ on the set X is the group of automorphisms of $T(X)$ generated by $\{\sigma_x \mid x \in X\}$.

Proposition 1.5. *The action of $F(X)$ on $T(X)$ satisfies the following properties:*

- *The labels of the directed edges are preserved.*

- *if an element g of $F(X)$ fixes a vertex or an undirected edge of $T(X)$ then g is the identity.*

Proof. Lemma 1.3 shows that labels are preserved. If g is an automorphism of $T(X)$ that preserves labels and fixes a vertex v, then g fixes all directed edges that have v as an endpoint. In particular, a geodesic cannot join a fixed vertex to a non-fixed vertex. Since $T(X)$ is connected, either every vertex is fixed by g, in which case g is the identity, or no vertex is fixed by g. If an undirected edge is fixed by an automorphism, but its endpoints are not fixed, then the two corresponding directed edges are interchanged. This contradicts the fact that labels are preserved. Therefore a non-identity element of $F(X)$ cannot fix an undirected edge. $\square$

Corollary 1.6. *The free group $F(X)$ is the group of label-preserving automorphisms of $T(X)$.*

1.7. Define a function $\mathcal{W}(X) \to F(X)$, as follows

$$\text{If } w = x_1 \cdots x_n \text{ then } w \mapsto \sigma_w \doteq \sigma_{x_1} \circ \cdots \circ \sigma_{x_n}.$$

If w is a reduced word, so that it is a vertex of $T(X)$, then the automorphism $\sigma_w \in \mathrm{Aut}(T(X))$ sends the vertex 1 to the vertex w. Thus the map $w \mapsto \sigma_w$ restricts to a bijection between the set of reduced words in $\mathcal{W}(X)$ and the elements of $F(X)$.

Define a symmetric relation $\sim$ on $W(X)$ by specifying that

$$u \diamond x \diamond \bar{x} \diamond v \sim u \diamond v,$$

whenever $x \in X \sqcup \bar{X}$ and $u, v \in \mathcal{W}(X)$. This is not an equivalence relation, since it is not transitive, but we can consider the the equivalence relation $\approx$ generated by $\sim$. In other words, we define $u \approx v$ if there exist words $u = w_0, \dots, w_n = v \in \mathcal{W}(X)$ such that $w_{i-1} \sim w_i$ for $i = 1, \dots n$.

Corollary 1.8. *If u and v are words in $\mathcal{W}(X)$ then $u \approx v$ if and only if $\sigma_u = \sigma_v$ in $F(X) \subseteq \mathrm{Aut}(T(X))$.*

To summarize, every word in $\mathcal{W}(X)$ determines a unique element of $F(X)$, and there is a $1-1$ correspondence between elements of $F(X) \subseteq \mathrm{Aut}(T(X))$ and reduced words in $\mathcal{W}(X)$ given by $g \leftrightarrow g(1)$. If we think of elements of $F(X)$ as being represented by words in $\mathcal{W}(X)$ then the multiplication operation of $F(X)$ is given by concatenating the words and cancelling. Any two sequences of cancellation operations must produce the same reduced word.

Corollary 1.9 (Universal property)**.** *Let G be a group. Any function $f : X \to G$ extends to a unique group homomorphism $\hat{f} : F(X) \to G$.*

Proof. If $w = x_1 \cdots x_n$ is a word in $\mathcal{W}(X)$, then we define $\hat{f}(\sigma_w) = f(x_1) \cdots f(x_n)$. Note that if $w \approx v$ then $\hat{f}(\sigma_w) = \hat{f}(\sigma_w)$. Since $\sigma_w = \sigma_v$ if and only if $w \approx v$, this shows that $\hat{f}$ is well-defined. □

2. Tree geometry

Tree geometry is so much fun that these facts are best left as exercises.

Exercise 2.1. Any two vertices v_1 and v_2 of a tree are joined by a unique geodesic.

Definition 2.1. The length of the geodesic joining v to w is the *distance* from v_1 to v_2, denoted $d(v_1, v_2)$.

Exercise 2.2. If δ is an edge-path from v to w and γ is the geodesic from v to w then every edge of γ is an edge of δ.

Exercise 2.3. The function d is a metric. In particular, the vertices of a tree form a metric space with distance function d.

Exercise 2.4. The intersection of two subtrees of a tree is a tree.

Exercise 2.5. If U is a subtree of a tree T and v is any vertex then there is a unique vertex of U which is closest to v.

Exercise 2.6. If v_1 and v_2 are two vertices of a tree T then $\{v \mid d(v, v_1) \leq d(v, v_2)\}$ is a tree.

Exercise 2.7. The forest obtained by removing one edge of a tree has two components.

3. A characterization of free groups

Lemma 3.1 (Ping-Pong Lemma). *Let G be a group acting on a set S. Suppose there exist*

- *a set X of generators of G;*
- *a collection $\mathcal{S} = \{S_g \mid g \in X \cup X^{-1}\}$ of subsets of S; and*
- *a point $p \in S - \bigcup \mathcal{S}$*

such that

(1) $g(p) \in S_g$ for each $g \in X \cup X^{-1}$; and

(2) $g(S_h) \subseteq S_g$ for all $h \in X \cup X^{-1} - \{g^{-1}\}$.

Then $G \cong F(X)$.

Proof. By the universal property we have a surjective homomorphism $\phi : F(X) \to G$ such that $\phi(x) = x$ for all $x \in X$. It suffices to show that if $w \neq 1$ is a reduced word in $\mathcal{W}(X)$ then $\phi(\sigma_w) \neq 1$. To show that an element of G is not the identity we will show that it doesn't fix the point p. In fact, we will show by induction that if $w = x_1 \cdots x_n$ is a reduced word in $\mathcal{W}(X)$ and $g = \phi(\sigma_w)$ then $g(p) \in X_{x_1}$. This follows from the condition (1) in the case $n = 1$. For $n > 1$ we know by induction that if $h = \sigma_v$, where $v = x_2 \cdots x_n$, then $h(p) \in X_{x_2}$. Since $x_2 \neq \bar{x}_1$, we have $g(p) = x_1(h(p)) \in X_{x_1}$ by condition (2). $\square$

Definition 3.2. A group G of automorphisms of a tree T acts *freely* if no non-identity element of G fixes a vertex or an undirected edge of T.

Lemma 3.3. *If G acts freely on a tree T, and if g is a non-identity element of G then $g \neq g^-1$.*

Proof. Suppose $1 \neq g \in G$ and $g = g^{-1}$. Let v be any vertex of T. Consider the geodesic $\gamma = e_1 \cdots e_n$ joining v to $g(v)$. Since g interchanges the endpoints of γ, and the geodesic joining two points of a tree is unique, we must have $\gamma(e_i) = \bar{e}_{n-i}$. If n is even then γ fixes the vertex $\omega(e_{n/2}) = \alpha(e_{(n/2)+1})$. If n is even then γ fixes the unoriented edge $(e_{(n+1)/2}, \bar{e}_{(n+1)/2})$. In either case this is a contradiction to the assumption that G acts freely. $\square$

Definition 3.4. Suppose that a group G acts on a tree T. A *fundamental domain* for G is a subtree D of T such that $V(D)$ contains exactly one vertex from each G-orbit in $V(T)$.

Lemma 3.5. *If a group G acts freely on a tree T then there is a fundamental domain for G.*

Proof. We first use Zorn's Lemma to show that there exists a subtree D of T which is maximal in the family $\mathcal{F}$ of all subtrees which contain at most one vertex from each G-orbit. Any vertex of T is such a subtree of T, so the family is non-empty. Suppose $\mathcal{C} \subseteq \mathcal{F}$ is a chain. Let U denote the union of all of the subtrees in $\mathcal{C}$. Clearly U is connected, so it is a subtree. Suppose that v is a vertex and g is a non-identity element of G such that v and gv are both contained in U. Then g and gv are both contained in some subtree $C \in \mathcal{C}$, which is impossible since $C \in \mathcal{F}$. Thus $U \in \mathcal{F}$, so there exists a maximal subtree in $\mathcal{F}$.

If D does not contain a vertex from each G-orbit then $\cup_{g \in G} V(gD) \neq V(T)$. Since T is connected, there exists an edge e such that $\alpha(e) \in V(g_0 D)$ for some $g_0 \in G$, but $\omega(e)$ is not contained in $V(gD)$ for any $g \in G$. Let $f = g_0^{-1} e$, so $\alpha(f) \in V(D)$ but $w = \omega(f) \notin V(gD)$ for all $g \in G$. Let E be the subtree obtained by adding the edge f and the vertex w to D. Since D is maximal, there must exist a non-identity element h

of G so that $E \cap hE \neq \emptyset$. Since $V(hD) \cap V(D) = \emptyset$, the only vertex of E which could possibly be contained in the intersection is w. But $hw \notin V(D)$, so we must have $hw = w$. This is impossible since G acts freely. $\square$

Definition 3.6. Suppose that G acts freely on a tree T and that D is a fundamental domain for G. The directed edges e of T such that $\alpha(e)$ is a vertex of D, but $\omega(e)$ is not a vertex of D, will be called the *boundary edges* of D. The set of boundary edges of D will be denoted ∂D.

Theorem 3.7. *A group is free if and only if it acts freely on a tree.*

Proof. By construction any free group acts freely on a tree, so we must only prove the other implication.

Assume that G acts freely on a tree T. Let D be a fundamental domain for G. For each edge e in ∂D there exists a unique element $g_e \in G$ such that $\omega(e) \in V(g_e D)$. Set $S = \{g_e \mid e \in \partial D\}$. If $e \in \partial D$ then $f = g_e^{-1}\bar{e} \in \partial D$, and $g_f = g_e^{-1}$. Thus $S = S^{-1}$.

Next we will show that S generates G. Fix a vertex v of D. Let $B \subseteq E(T)$ be the union of the G-orbits of edges in ∂D. For each $g \in G$, let $b(g)$ denote the number of edges in the geodesic from v to gv which lie in B. Set $G_n = \{g \in G \mid b(g) = n\}$. We will show by induction on n that $G_n \subseteq \langle S \rangle$. Observe that $G_1 = S$. Assume that $G_n \subseteq \langle S \rangle$ and let $g \in G_{n+1}$. Consider a geodesic γ from v to gv. The geodesic γ must contain (exactly) one edge e of ∂D. Thus $\gamma = \gamma_1 \star \gamma_2$ where γ_1 contains e and γ_2 joins $g_e v$ to gv. Since $|\gamma_2| = n$ and since $g_e^{-1}\gamma_2$ joins v to $g_e^{-1}gv$, we have $g_e^{-1}g \in \langle S \rangle$. Since $g_e \in S$, it follows that $g \in \langle S \rangle$.

We are now ready to apply the ping-pong lemma. By Lemma 3.3, no non-identity element of G is equal to its inverse. Thus we may write $S = X \sqcup X^{-1}$. For $e \in \partial D$, define S_{g_e} to be the set of all vertices w of T such that the geodesic from v to w contains the edge e. Set $p = v$. If e is an edge in ∂D then the geodesic from v to $g_e D$ contains e, since (e) is the geodesic from D to $g_e D$. Thus condition (1) of the ping-pong lemma is satisfied. To verify condition (2), let $g = g_e \in X \sqcup X^{-1}$, and suppose that $w \in S_h$ where $h = g_f \neq g_e^{-1}$. This means, in particular that $g_e^{-1}(\bar{e}) \neq f$. The edge $g_e^{-1}(\bar{e})$ joins $g_e^{-1}D$ to D. Consider the edge-path $\gamma = \gamma_1 \star \gamma_2 \star \gamma_3$ where γ_1 is the geodesic from $g_e^{-1}(v)$ to $\omega(g_e^{-1}e) \in D$, γ_2 is the geodesic from $\omega(g_e^{-1}e)$ to $\alpha(f) \in D$ and γ_3 is the geodesic from $\alpha(f)$ to w. Since γ_2 is contained in D while $g_e^{-1}(e) \neq \bar{f}$, it follows that γ is a geodesic joining $g_e^{-1}v$ to w. The geodesic $g_e\gamma$ joins v to $g_e w$ and contains the edge e. This shows that $g_e w \in S_{g_e}$, verifying condition (2) of the ping-pong lemma. $\square$

Corollary 3.8 (Nielsen-Schreier Theorem)**.** *Any subgroup of a free group is free.*

4. Group presentations

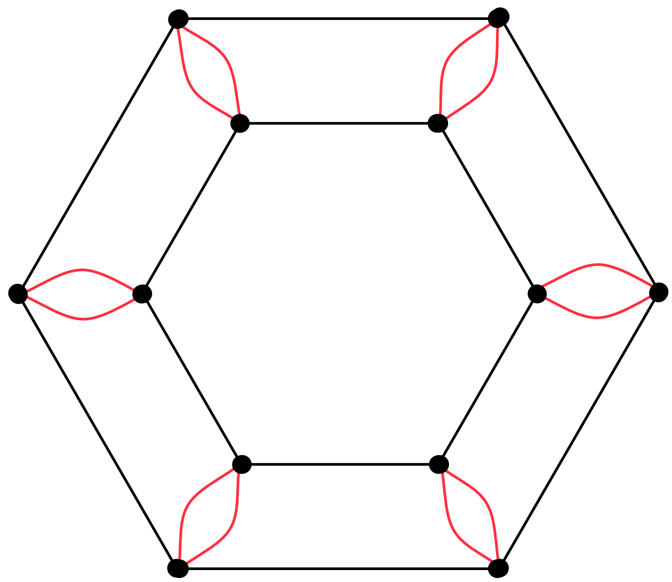
Suppose that G is a group generated by a set $X = \{x_1, \ldots\}$ of elements of G. By the universal property of free groups, the inclusion map from X to G extends to a homomorphism $\phi : F(X) \to G$ which is surjective since X generates G. Suppose that $R = \{r_1, \ldots\}$ is a collection of elements of $F(X)$ which normally generate $\ker\phi$; that is, suppose that $\ker\phi$ is the smallest normal subgroup of G which contains R. Then $|X : R|$ is a *presentation* of G. The elements of X are called *generators* and the elements of R are called *relators*. Sometimes the relators are replaced by *relations* which are equations. The relator r is replaced by the equation $r = 1$, or by an equivalent equation. For example, the relator $xyx^{-1}y^{-1}$ might be replaced by the relation $xy = yx$.

Example 4.1. The dihedral group D_{2n} is generated by a rotation r of order n and a reflection s of order 2 which satisfy the equation $srs = r^{-1}$. Every element of the dihedral group can be uniquely written as r^is^j, where $0 \le i \le n-1$ and $0 \le j \le 1$. Let $X = \{r, s\}$ and let $\phi : F(X) \to D_{2n}$ denote the surjective homomorphism which extends the inclusion map of X into D. The relators r^n, s^2 and $srsr$ are all contained in $\ker\phi$. Thus, if we let N denote the normal subgroup of $F(X)$ which is normally generated by these three relators, then we have $N \le \ker\phi$. On the other hand, it is easy to check that any word in $F(\{r, s\})$ is equivalent, modulo N, to a word of the form r^is^j, where $0 \le i \le n-1$ and $0 \le j \le 1$. That is, every element of $F(\{r, s\})$ can be written as a product of a word of this form and an element of N. This shows that $\ker\phi \le N$, so the dihedral group of order $2n$ has a presentation

$$D_{2n} = |s, r : s^2, r^n, srsr| = |s, r : s^2 = r^n = 1, srs = r^{-1}|.$$

4.2. Suppose that N is a normal subgroup of $F(X)$. We can then consider the action of N by label-preserving automorphisms of the tree $T(X)$. There is only one orbit of vertices under the action of $F(X)$, so the N-orbits of vertices are in one-to-one correspondence with the cosets of N, i.e. with the elements of the quotient group $G = F(X)/N$. Since the action of N preserves labels, the orbits of edges also inherit labels. The orbits of the vertices and the orbits of the edges can be regarded in a natural way as vertices and edges of a graph, which is denoted $T(X)/N$, and is called the *Cayley graph* of G. The vertices of the Cayley graph can be identified with the elements of G. The directed edges of the Cayley graph have labels, which are elements of the generating set X. A vertex g of $T(X)/N$ is joined to a vertex h by an edge with label x if $gx = h$, where g, h and x have been identified with elements of G in the natural way.

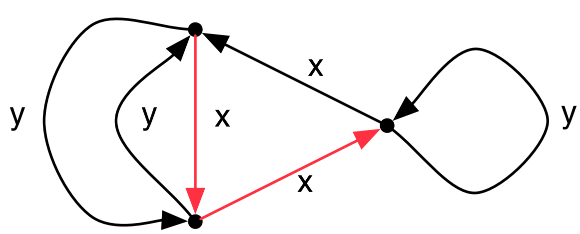
Here is a picture of the Cayley graph for D_{12}.



The Cayley graph of D_{12}

4.3. If G is any subgroup of $T(X)$, not necessarily normal, we may still construct the quotient graph $\Gamma = T(X)/G$. The vertices will correspond to the cosets of G in $F(X)$, but these cosets will not form a group. The graph does, however, provide a description of the subgroup G, up to conjugation by an arbitrary element of $F(X)$. (Conjugate subgroups of $F(X)$ will have isomorphic quotient graphs.)

The fundamental domain D maps injectively into Γ, and the image S is a tree in Γ which contains every vertex. (Such a tree is called a *spanning tree* for the graph Γ. The edges of Γ which are not contained in S are in one-to-one correspondence with a set of free generators of (some conjugate of) G and their inverses. Moreover, up to conjugation of the entire subgroup G, one can construct the words that represent these generators using the labels of the edges of Γ. Fix a base vertex $v \in V(S)$. Let e be a directed edge in Γ which is not contained in S, and let γ_1 and γ_2 be geodesics from v to $\alpha(e)$ and $\omega(e)$ respectively. Then $\gamma_1 \star (e) \star \bar{\gamma}_2$ is a geodesic which joins v to v. The word that represents g_e is obtained by reading the labels of the directed edges of this geodesic, since this is the same word that would be obtained by reading the labels of the geodesic from v to $g_e v$ in the tree $T(X)$.



The picture above shows the quotient graph $\Gamma = T(\{x, y\})/G$, where G is the subgroup of G which is freely generated by the elements xyx^{-1}, xxx, y, and $x^{-1}y$ of $F(\{x, y\})$. Since there are three vertices, one can see that G has index 3 in $F(\{x, y\})$. The tree S is shown in red.

Exercise 4.1. Show that if N is a normal subgroup of $F(X)$ and $\Gamma = T(X)/N$ then the quotient group $G = F(X)/N$ acts freely on Γ by label-preserving symmetries.

Exercise 4.2. Show that a finitely generated normal subgroup of a free group must have finite index.