Some examples:

Fundamental groups of graphs: Every finite connected graph Γ has a *maximal tree* T, a connected subgraph with no simple circuits. Since any tree is the union of smaller trees joined at a vertex, we can, by induction, show that $\pi_1(T) = \{1\}$. In fact, if e is an outermost edge of T, then T deformation retracts to $T \setminus e$, so, by induction, T is contractible. Consequently (*Hatcher, Proposition 0.17*), Γ and the quotient space Γ/T are homotopy equivalent, and so have the same π_1. But $\Gamma/T = \Gamma_n$ is a bouquet of n circles for some n. If we let $\mathcal{U}$ = a neighborhood of the vertex in Γ_n, which is contractible, then, by singling out one petal of the bouquet, we have

$$\Gamma_n = (\Gamma_{n-1} \cup \mathcal{U}) \cup (\Gamma_1 \cup \mathcal{U}) = X_1 \cup X_2$$

with $\Gamma_k \cup \mathcal{U} \sim (\Gamma_k \cup \mathcal{U})/\mathcal{U} \cong \Gamma_k$. Since $X_1 \cap X_2 = \mathcal{U} \sim *$, we have

$$\pi_1(\Gamma_n) \cong \pi_1(\Gamma_{n-1}) *_1 \pi_1(\Gamma_1) = \pi_1(\Gamma_{n-1}) * \mathbb{Z}$$

So, by induction, $\pi_1(\Gamma) \cong \pi_1(\Gamma_n) \cong \mathbb{Z} * \cdots * \mathbb{Z} = F(n)$, the free group on n letters, where n = the number of edges not in a maximal tree for Γ. The generators for the group consist of the edges not in the tree, prepended and appended by paths to the basepoint.

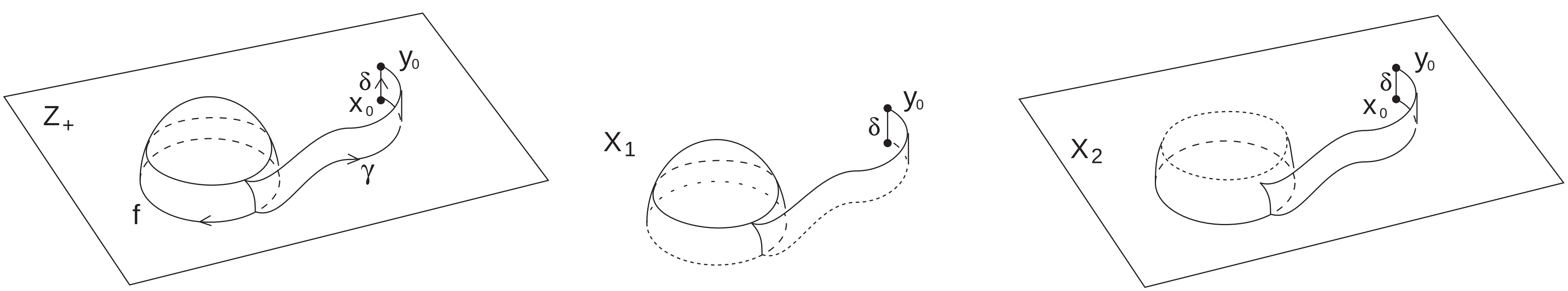
Gluing on a 2-disk: $f : \partial\mathbb{D}^2 \to X$ continuous, then we construct the quotient space $Z = (X \coprod \mathbb{D}^2)/\{x \simeq f(x) : x \in \partial\mathbb{D}^2\}$, the result of gluing $\mathbb{D}^2$ to X along f.

We can use Seifert - van Kampen to compute π_1 of the resulting space; if we wish to be careful with basepoints x_0, we include a rectangle R, the mapping cylinder of a path γ running from $f(1,0)$ to x_0, glued to $\mathbb{D}^2$ along the arc from $(1/2,0)$ to $(1,0)$ (see figure). This space Z_+ deformation retracts to Z; it is simpler work with the basepoint y_0 lying above x_0.

Write $D_1 = \{x \in \mathbb{D}^2 : ||x|| < 1\} \cup (R \setminus X)$ and $D_2 = \{x \in \mathbb{D}^2 : ||x|| > 1/3\} \cup R$, then we can write $Z_+ = D_+ \cup (X \cup D_2) = X_1 \cup X_2$. But since $X_1 \simeq *$, $X_2 \simeq X$ (it is essentially the mapping cylinder of the maps f and γ) and $X_1 \cap X_2 = \{x \in \mathbb{D}^2 : 1/3 < ||x|| < 1\} \cap (R \setminus X) \simeq S^1$, we find that

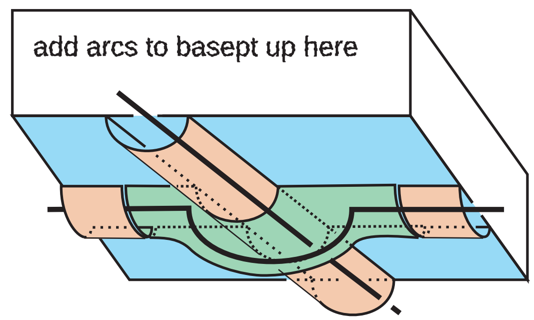
$$\pi_1(Z, y_0) \cong \pi_1(X_2, y_0) *_{\mathbb{Z}} \{1\} = \pi_1(X_2)/ < \mathbb{Z} >^N \cong \pi_1(X_2)/ < [\overline{\delta} * \overline{\gamma} * f * \gamma * \delta] >^N$$

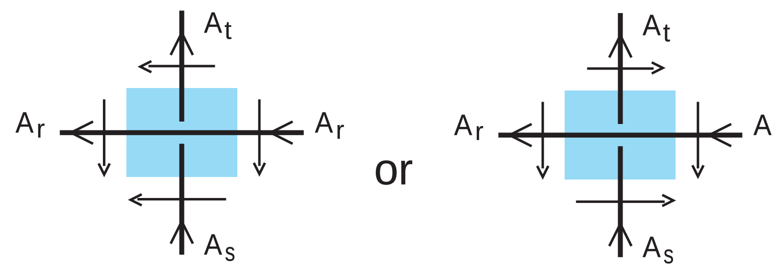
Use δ for a change of basepoint isomorphism, and then a homotopy equivalence from X_2 to X (fixing x_0), we have, if $\pi_1(X, x_0) =< \Sigma | R >$, then $\pi_1(Z) =< \Sigma | R \cup \{[\overline{\gamma} * f * \gamma]\} >$. So the effect of gluing on a 2-disk on the π_1 is to add a new relator, namely the word represented by the attaching map (adjusting for basepoint).



Wirtinger presentations for knot complements:

A *knot* K is (the image of) an embedding $h : S^1 \hookrightarrow \mathbb{R}^3$. Wirtinger gave a prescription for taking a planar projection of K and producing a presentation of $\pi_1(\mathbb{R}^3 \setminus K) = \pi_1(X)$. The idea: think of K as lying on the projection plane, except near the crossings, where it arches under itself. We build a CW-complex $Y \subseteq X$ that X deformation retracts to. A presentation for $\pi_1(Y)$ gives us $\pi_1(X)$.

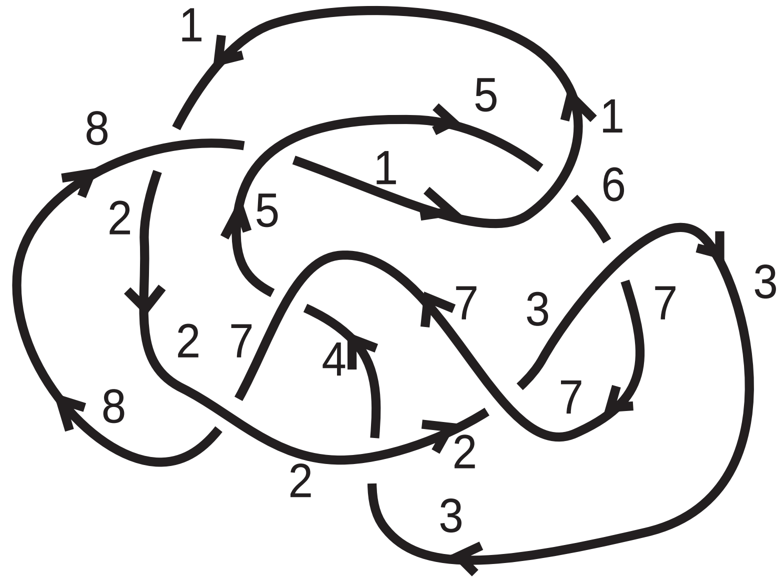




To build Y, glue rectangles arching under the strands of K to a horzontal plane lying just above the projection plane of K. At the crossing, the rectangle is glued to the rectngle arching under the over-strand. X deformation retracts to Y; the top half of $\mathbb{R}^3$ deformation retracts to the top plane, the parts of X inside the tubes formed by the rectangles radially retract to the boundaries of the tubes, and the bottom part of X vertically retracts onto Y. Formally, we should really keep a "slab" above the plane, to give us a place to run arcs to a fixed basepoint in the interior of the slab.

We think of Y as being built up from the slab C, by gluing on annuli $A_i \cong S^1 \times I$, one for each rectangle R_i glued on; the rectangle S_i lying above R_i in the bottom of the slab C is the other half of the annulus. Then we glue on the 2-disks D_j, one for each crossing of the knot projection. A little thought shows that there are as many annuli as disks; the annuli correspond to the unbroken strands of the knot projetion, which each have two ends, and each crossing is where two ends terminate (so there are two ends for every A_i and two ends for every D_j, so there are half as many of each as there are total number of ends). To make sure that all of our interections are path connected, and to formally use a single basepoint in all of our computations, we join every one of the annuli and disks to a basepoint lying in the slab by a collection of (disjoint) paths.

Now starting with the slab (with $\pi_1 = 1$), add the A_i one at a time; each has $\pi_1 = \mathbb{Z}$, generated by a loop which travels once around the S^1-direction, and its intersection with $C\cup$ the previously glued on annuli is the rectangle S_i, which is simply connected. So, inductively, $\pi_1(C\cup A_1\cup\cdots\cup A_i) \cong \pi_1(C\cup A_1\cup\cdots\cup A_{i-1})*\pi_1(A_i) \cong F(i-1)*\mathbb{Z} \cong F(i)$ is the free group on i letters, so, adding all n (say) of the annuli yields $F(n)$. Then we glue on the n 2-disks D_j; these add n relators to the presentation $\langle x_1, \ldots , x_n | \rangle$. To determine the relators, choose specific generators for our $\pi_1(A_i)$, by *orienting* the knot (choosing a direction to travel around it) and choosing the loop which goes counter-clockwise around the annulus, when you face in the direction of the orientation. Going around the boundary of the 2-disk D_j spells out the word $x_r x_s x_r^{-1} x_t^{-1}$ or $x_r x_s^{-1} x_r^{-1} x_t$ reading counter-clockwise, depending on orientations. Carrying this out for every 2-disk completes the presentation of $\pi_1(Y) \cong \pi_1(X)$.



With practice, it becomes completely routine to read off a presentation for the fundamental group of $\mathbb{R}^3 \setminus K$ from a projection of K. For example, from the projection above, we have

$$\pi_1(\mathbb{R}^3 \backslash K) \cong \langle x_1, \ldots, x_8 | x_8x_1 = x_2x_8, x_2x_7 = x_8x_2, x_5x_8 = x_1x_5, x_1x_5 = x_6x_1, x_3x_6 = x_7x_3, x_7x_2 = x_3x_7, x_3x_2 = x_2x_4, x_7x_4 = x_5x_7\rangle$$