The Alexander polynomial is symmetric: $\Delta_K(t) = \Delta_k(t^{-1})$
following R.H. Fox and G. Torres [Ann. Math. **59** #2 (3-1954) 211-218]

In previous lectures we have shown that if we define the (single-variable) Alexander for oriented knots and links, then the effect of reversing the orientations on all components is, up to multiplication by $\pm t^n$, to replace the variable t with its inverse t^{-1}. We now show that the Alexander polynomial itself is invariant (up to the same multiplications by units) under this transformation. We shall do this using the Fox derivative approach to $\Delta_K(t)$.

Fox's approach begins with a presentation $G =< A|R >$ for the fundamental group $\pi_1(S^3 \setminus K)$ of the complement of a knot or link K, together with a choice of surjective homomorphism $G \to \mathbb{Z}$. For consistency, we will make a canonical choice using the orientation of K, by insisting that a loop which circles the knot using the righthand rule (curling around the knot in the direction of our fingers when the thumb is aligned with the orientation) is mapped to the generator t, and a loop travelling the opposite direction is mapped to t^{-1}. [These are basically the Wirtinger generators (and their inverses), which we noted previously are carried under abelianization to the generator or its inverse.]

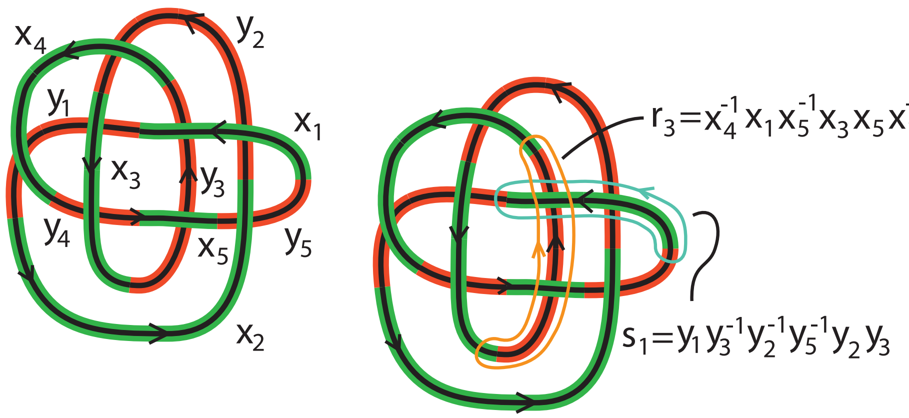
To compute $\Delta_K(t)$ we start by computing the *Fox derivatives* of the relators $r_j = x_{j_1}^{\epsilon_1} \cdots x_{j_k}^{\epsilon_k}$ with respect to each of the generators x_i, by accumulating terms in an (initially empty) sum as follows: reading r_j from left to right, each occurance of the generator x_i occurs as $r_j = ux_iv$ or $r_j = ux_i^{-1}v$. In the first case we add u, and in the second case we add $-ux_i^{-1}$. The resulting sum should be thought of as an element in the group ring of the free group $F(X)$ generated by the symbols X, and is denoted $\frac{\partial r_j}{\partial x_i}$. The canonical homomorphism ψ from $F(X)$ to G (sending each x_i to x_i), followed by our chosen homomomorphism ϕ from G to $\mathbb{Z} =< t| >$, induces ring homomorphisms between their group rings; their composition sends each $\frac{\partial r_j}{\partial x_i}$ to a (Laurent) polynomial in t with coefficients in $\mathbb{Z}$.

We have seen that these polynomials, when assembled into a 'Jacobian' matrix

$$J = (\frac{\partial r_j}{\partial x_i})^{\psi\phi},$$

yields a matrix which depends upon the presentation for G chosen. But any two (finite) presentations of G can be transformed one to the other by Teitze transformations, and the effect on the Jacobian matrix under these moves can be codified. In particular, the resulting moves on the Jacobian do not change (for J an $n \times m$ matrix) the *ideal (in* $\mathbb{Z}[\mathbb{Z}]$*) generated by the* $(m-k) \times (m-k)$ *minor determinants* of J. In particular, taking the ideal I_1 generated by the $(m-1) \times (m-1)$ minors, I_1 is principal, and a generator for I_1 is the Alexander polynomial $\Delta_K(t)$ (well defined up to multiplication by a unit $\pm t^n$ in $\mathbb{Z}[\mathbb{Z}]$).

To prove that the Alexander polynomial is symmmetric, Fox and Torres construct a pair of 'dual' presentations for G, as follows: Give a diagram of the knot/link K, partition the loop representing K into alternating 'oversegments' and 'undersegments': an oversegment passes over every crossing in the diagram that it meets, and an undersegment passes under. For example, in the Wirtinger presentation we take the oversegments to be the entire segment running between successive undercrossings as we traverse the knot using the orientation, and the undersegments are the remaining short segments passing under each crossing. The figure below shows a more effecient way to do this for the knot 8_{20}.



The basic idea is that an over/under representation of K enables us to build two 'dual' presentations of G, which we will use to establish our symmetry result. The idea is that the oversegments give generators of a presentation for G and the undersegments provide the relators for the presentation; the opposite is true for the second presentation. The idea is really the same as for the Wirtinger presentation: choosing our nose as the basepoint for $\pi(S^3 \setminus K)$, each oversegment corrresponds to a loop which runs from our nose to a point just to the right of the segment (using the orientation on K to determine 'right'), passes under the oversegment, and then directly back to our nose. The relators are determined by drawing a loop around each undersegment and which closely follows it. From a fixed starting point we will cross a sequence of oversegments; reading off this sequence (thinking of ourselves as actually travelling slightly below the plane of the knot projection) reads off a word in the generators which is our relator. To be precise about things, we will orient the loops around the undersegments to run counterclockwise around and oversegment, and clockwise around an undersegment. The proof that G is presented with generators given by the loops around oversegments and relators given by reading off the words surrounding undersegments is essentially identical to that for the Wirtinger presentation; in that case the loop under and surrounding a crossing is identical to the loop surrounding the undersegment. [Of course, we did not present that proof, but any discussion of the Wirtinger presentation can be adapted with no change to this more general setting.

Oversegments and loops around undersegments therefore gives us one presentation. (Note that we, mostly for convenience, label the segments around the projection as $x_1, y_1, x_2, y_2, \ldots$.) The 'dual' presentation comes essentially from changing our point of view to behind the projection plane. Then what we have called undersegments become oversegments and vice versa. So we can apply the same argument to build a second presentation. We will however retain a viewpoint from the front of the projection plane, in order to discuss both presentation at the same time. The 'under' presentation will have generators y_j corresponding to undersegments; the specific loop representing the generator we will think of as running from right to left across the undersegment, but this loop is now starting at a point lying behind the projection plane, and passing over the strand. **The important point:** as a result, when we map to $\mathbb{Z}$ using the orientation, each y_j will map to t^{-1} (and not t). The relators for the underpresentation come from reading the words in the y_j that the loops running around the oversegments (and thought of as lying slightly above the projection plane) spell out. It is essentially because our basepoint for the underpresentation lies behind the projection plane that we orient the loops around the oversegments in the opposite direction to the loops around undersegments for the overpresentation: from the perspective of the basepoint, we are really reading around the loops in the same directions.

The basic point here is that the generators for the overpresentation come from the oversegments and the relators come from the undersegments. For the underpresentation it is the exact opposite: the generators come from the undersegments and the relators come from the oversegments. Our plan then is to compare computations of $\frac{\partial r_j}{\partial x_i}$ and $\frac{\partial s_i}{\partial y_j}$, and show that they are identical (after replacing t with t^{-1}). This will show that the Jacobian matrices for the two presentations (after changing the base rings to $\mathbb{Z}[\mathbb{Z}]$) satisfy $(J_{\text{over}})^{\psi\phi}(t) = [(J_{\text{under}})^{\psi\phi}(t^{-1})]^T$. Therefore, since a matrix and its transpose have the same determinant, (after changing the variable) the ideals generated by the codimenskion-1 minor determinants are identical, and therefore have the same generator, i.e., $\Delta_K(t) = \Delta_K(t^{-1})$, as desired.

Each of the two Fox derivatives $\frac{\partial r_j}{\partial x_i}$ and $\frac{\partial s_i}{\partial y_j}$ is a formal sum, over the specific instances of the variable x_i (resp. y_j) in the relator r_j (resp x_i). These instances occur in one of three ways, which we will treat in turn. The idea is that each instance can be paired with a corresponding instance in the other Fox derivative, and that the contributions to the Fox derivatives of each is identical once we replace t with t^{-1} in one of the two computations.

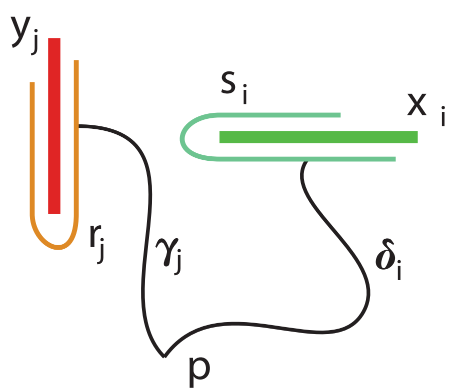
First, note that if the oversegment x_i and the undersegment y_j do not meet, then x_i does not appear in r_j and y_j does not appear in s_i, so both Fox derivative are 0, and so are equal (after replacing t in one with t^{-1}...!). So in what follows we need only concern ourselves with cases where the two segments (and so, in particular, the two loops r_j and s_i) intersect.

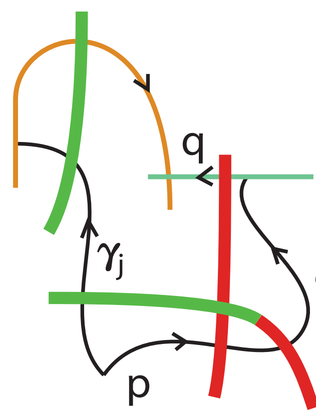
The first instance, which we did not explore in class, is the fact that the loops representing r_j and s_i must actually represent based loops, using a fixed and common basepoint (one for all of the r_j, another for all of the s_i). This is in fact the **single greatest pitfall** in carrying out fundamental group computations; we must use a single basepoint throughout the computation, which in essence really means that for a loop such as the r_j and s_i we wish to use, we should read the word starting at a fixed point along the loop, and must pre- and post-pend a path to a fixed and common basepoint. (Failure to do so means that a word we might be using is actually a conjugate of the word we really should be using.) But this is not nearly so difficult an issue with the Fox calculus as it might seem. Because of the fact that the Fox derivative satisfies (using the shorthand notation u_x for the Fox derivative of u w.r.t. x) $(uv)_x = u_x + u(v_x)$, we find that for a relator r,

$$(uru^{-1})_x = u_x + u(r_x) + ur(u^{-1})_x = u_x + ur_x - uru^{-1}u_x$$

But when we push forward from $F(X)$ to $\mathbb{Z}$, every relator r is sent to (1 in G, and so is sent to) 1 in $\mathbb{Z}$, so uru^{-1} is sent to $uu^{-1} = 1$, so $(uru^{-1})_x = u(r_x)$. The effect of conjugation on the Fox derivative of a relator, that is, the effect of the choice of path to the basepoint, is therefore to multiply by the (image in $\mathbb{Z}$ of the) conjugator (on the left).

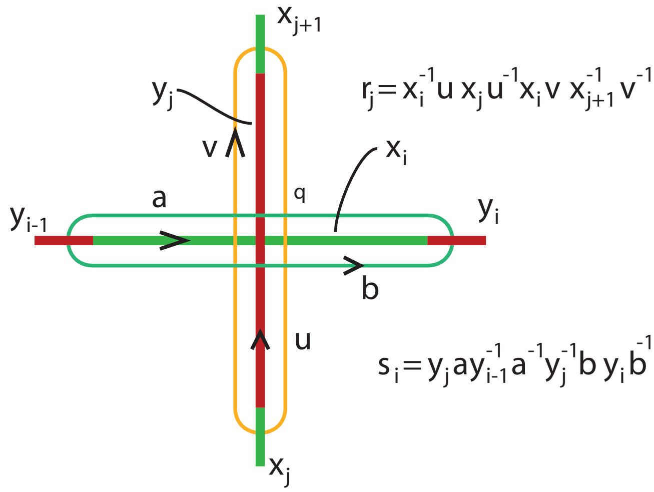
For our comparison of Fox derivatives, we choose fixed basepoints lying above/below the same point p in the projection plane and away from the knot projection, and choose fixed paths γ_j, δ_i from p to each of the loops r_j and s_i lying slightly below/above the projection plane; the paths γ_j should avoid the undersegments (and therefore pass under the oversegments), and the δ_i should similarly avoid the oversegments. [This is possible because, in the projection plane, the complement of the set of oversegments is connected (and the same for the collection of undersegments); this is most easily seen by induction on the number of segments.]





But! We really would like to use a basepoint q lying at the intersection of r_j and s_i (for reasons that will become apparent). That is, in our computations and comparisons, we would like to simply read around the relator starting from a common point of intersection. So we move the endpoints of γ_j and δ_i, by appending a segment of the loops r_j and s_i, to have these loops start and end at q. (Essentially, we are writing $\gamma_j r_j \gamma_j^{-1}$ as $\gamma_j uv\gamma_j^{-1} = [\gamma_j u]vu[u^{-1}\gamma_j^{-1}] = (\gamma_j u)vu(\gamma_j u)^{-1}$; these two words are equal in the free group and so have the same Fox derivative.) But now these appended paths, which we will still call γ_j and δ_i, begin and end at the same point in the projection plane, and so form a loop η. The γ_j half of this loop misses the undersegments, and the δ_i half misses the oversegments. Therefore, since the points of intersection of the knot itself with the loop η must pair up (by following the segments around the knot projection), the net number of times that each of γ_j and δ_i cross their respective segments *from right to left* must be the same; either the knot recrosses the two paths at points with the same 'color', in which case it must pass across the same path in opposite directions (yielding a net contribution of $t^0 = 1$ in $\mathbb{Z}[\mathbb{Z}]$), or the knot recrosses with the opposite color, in which case it crosses the other path in the same direction, pairing up the two crossings. [These facts are apparent from the figure: to establish them formally we can argue that if the parity were opposite then we could (from K and the paths) construct a loop in the projection plane that intersects the projection of K an odd number of times, which is impossible.]

The net effect is that the exponent sum of the x's in γ_j must equal the exponent sum of the y's in δ_i. If this common exponent sum is n, then conjugating r_j by γ_j multiplies the Fox derivative by t^n, and conjugating s_i by δ_i multiplies the Fox derivative by t^{-n} (since each of the y_j is sent to t^{-1}). Thus to take into account the path to a more conveniently chosen basepoint, each Fox derivative changes in a way consistent with our goal; if the terms of the derivative are equal (after variable change), then the conjugated terms are, as well. So in what follows we can ignore the effect of the path from the basepoint, and act as if the basepoint lies where we want it, at the intersection of the two loops r_j and s_i.



The most typical instances of the x_i, y_j in our relators occur where the oversegment x_i crosses the undersegment y_j. Each such crossing generates a pair of instances of x_i in r_j and a pair of instances of y_j in s_i. Choosing our basepoint q to lie at the intersection of the two loops as indicated in the figure above, we can read off the relators as

$$r_j = x_i^{-1}ux_ju^{-1}x_ivx_{j+1}^{-1}v^{-1} \text{ and } s_i = y_jay_{i-1}^{-1}a^{-1}y_j^{-1}by_ib^{-1}$$

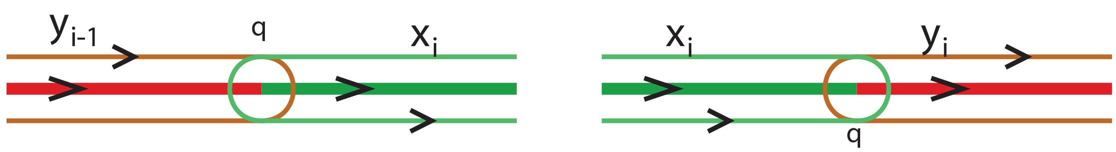
If we then compute the contributions to the Fox derivatives of each of the two relators, we find

For r_j : $-x_i^{-1} + x_i^{-1}ux_ju^{-1}$ and for s_i : $1 - y_jay_{i-1}^{-1}a^{-1}y_j^{-1}$

Then sending the x's to t and the y's to t^{-1}, u is sent to t^m and a is sent to t^m, so we get the contributions:

For $r_j : -t^{-1} + t^{-1}t^n t t^{-n} = 1 - t^{-1}$ and for $s_i : 1 - t^{-1}t^m t t^{-m} t = 1 - t$

So the contribution, as a function of t, to $\frac{\partial r_j}{\partial x_i}$ is equal to the contribution to $\frac{\partial s_i}{\partial y_j}$, evaluated at t^{-1}.



The final contribution comes from the fact that r_j contains instances of x_j and x_{j+1}, and s_i contains instances of y_{i-1} and y_i. So if $i = j$, or if $i = j + 1$ (so $j = i - 1$), each has a single additional instance of the variable we are computing the Fox derivative with respect to. These occur precisely when the segments we are using follow one after the other, as in the figure above. In these cases, choosing our basepoint for comparison to be the point q depicted in the figure, the contribution of the remaining instance of each variable can be computed simply as

$i = j + 1$: $\frac{\partial r_j}{\partial x_i}$ has an additional $-x_i^{-1}$, evaluating to $-t^{-1}$, while $\frac{\partial s_i}{\partial y_j}$ has an additional $-y_{i-1}^{-1}$, evaluating to $-t$;

$i = j$: $\frac{\partial r_j}{\partial x_i}$ has an additional x_i, evaluating to t, while $\frac{\partial s_i}{\partial y_j}$ has an additional y_i, evaluating to t^{-1}.

As before, the first is then equal to the second, evaluated at t^{-1}.

Therefore, we find that for these dual presentations, the Jacobian matrices satisfy

$$(J_{\text{over}})^{\psi\phi}(t) = [(J_{\text{under}})^{\psi\phi}(t^{-1})]^T.$$

As remarked above, this implies that the Alexander ideals have generators that are equal to one another, after the variable t in one is replaced with t^{-1}. Since they are also equal to one another (being the result of the Fox calculus applied to two presentations of isomorphic groups), we conclude that

$$\Delta_K(t) = \Delta_k(t^{-1}),$$

up to multiplication by units $\pm t^n$ in the group ring $\mathbb{Z}[\mathbb{Z}]$.