BRAID ORDERING AND KNOT GENUS

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Abstract. The genus of knots is a one of the fundamental invariant and can be seen as a complexity of knots. In this note, we give a lower bound of genus using Dehornoy floor, which is a measure of complexity of braids in terms of braid ordering.

1. Introduction

Let $B_n = \langle \sigma_1, \cdots, \sigma_{n-1} | \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \ |i-j| = 1, \ \sigma_i\sigma_j = \sigma_j\sigma_i \ |i-j| \geq 2 \rangle$ be a braid group of n-strands. A braid $\beta \in B_n$ is called *σ-positive* if β can be represented by a braid word which contains at least one σ_i and contains no $\sigma_1^{\pm 1}, \sigma_2^{\pm 1} \cdots, \sigma_{i-1}^{\pm 1}, \sigma_i^{-1}$ for some $1 \leq i \leq n-1$. We say $\alpha < \beta$ is true if and only if braid $\alpha^{-1}\beta$ is σ-positive. It is known that relation $<$ defines total ordering of B_n which is invariant under left multiplication of B_n: That is, if braids $\alpha, \beta \in B_n$ satisfy $\alpha < \beta$, then for all braids $\gamma \in B_n$, $\gamma\alpha < \gamma\beta$ holds. We call this left-invariant total ordering *Dehornoy ordering*. There are many other interpretations and equivalent definitions of Dehornoy floor in both algebraic and geometric way (See [DDRW], excellent survey of this topic). Thus, the Dehornoy ordering is a quite natural structure of braid groups B_n.

Using Dehornoy ordering, we define *Dehornoy floor* $[\ ]_D$, which is a measure of complexity of braids using Dehornoy ordering, as follows. Let $\Delta = (\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_1\sigma_2\cdots\sigma_{n-2})\cdots(\sigma_1\sigma_2)(\sigma_1)$ *Garside's fundamental braid*. The Garside's fundamental braid has many special properties and plays important role in Braid group. For example, the center of braid group is an infinite cyclic group which is generated by Δ^2.

Definition 1. *Dehornoy floor $[\beta]_D$ of braid $\beta \in B_n$ is a minimal non-negative integer m which satisfies $\beta \in (\Delta^{-2m-2}, \Delta^{2m+2})$, where $(\Delta^{-2m-2}, \Delta^{2m+2}) = \{\alpha \in B_n \mid \Delta^{-2m-2} < \alpha < \Delta^{2m+2}\}$.*

We must be careful that when we think about Dehornoy floor, which B_n a braid β belongs to is very important. For example, for a braid

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$\beta = (\sigma_1\sigma_2)^4$, $[\beta]_D = 1$ if we consider $\beta \in B_3$, and $[\beta]_D = 0$ if we consider $\beta \in B_4$. Fortunately, if we use braid groups to describe links, the number of strands are always implicit, so there might be no confusion about the number of braid strands. So in most cases, we omit to write which B_n a braid β belongs to. Dehornoy floor is first appeared in [MN], though they do not use the term "Dehornoy floor". In this paper, they show Dehornoy floor can be seen as a restriction of admissibility of braid moves. That is, if a closure of a braid β admits some braid moves such as destabilization, which is defined by $A\sigma_n \mapsto A$, then Dehornoy floor of β is bounded. In our previous work [I], we prove that Dehornoy floor of braids gives some information about the position of essential surface in closed braid complements, so gives some geometric information of links represented by their closures. These works seems to suggest that there exist more unknown relationships between braid ordering and knot theory. The main purpose of this paper is to compare Dehornoy floor, the fundamental complexity of braids via Dehornoy ordering, and the genus of its closure, which is the most fundamental complexity of knots via topology.

Our main result is following:

Theorem 1. *Let $\beta \in B_n$ be a braid and $\chi(\widehat{\beta})$ be a maximal Euler characteristics of an orientable spanning surface whose boundary is $\widehat{\beta}$. Then, inequality*

$$[\beta]_D < 2 - \frac{2\chi(\widehat{\beta})}{n+2}$$

holds.

As a consequence, we obtain relationships between knot genus and Dehornoy floor.

Corollary 1. *Let K be an oriented knot and $g(K)$ be its genus. If a closure of a braid $\beta \in B_n$ is K, then*

$$[\beta]_D < \frac{4g(K)}{n+2} - \frac{2}{n+2} + 2$$

holds.

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2. Preliminaries

In this section, we prepare some of basic facts of braid ordering and braid foliation theory which will be used to prove theorem 1.

2.1. **Property of Dehornoy floor.** First we review following proposition which is proved in [I].

Proposition 1 ([I])**.** *If a braid β is conjugate to another braid represented by a braid word which contains s occurrence of σ_1 and k occurrence of σ_1^{-1}, then $[\beta]_D < max\{s,k\}$.*

Proposition 1 allows us to estimate Dehornoy floor of a braid using its representing word.

For later use, we prove a slightly different estimation of Dehornoy floor using band generator. For $1 \leq i < j \leq n$, let $a_{i,j}$ be a braid defined by

$$a_{i,j} = \sigma_j \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \cdots \sigma_j^{-1}.$$

The braids $a_{i,j}$ can be seen as a boundary of twisted band attached to i-th and j-th strands of braid and called *band generator*. We denote a monoid which is generated of $\{a_{i,j} | 1 \leq i < j \leq n\}$ by $B^+_{n\ band}$. The algebraic property of Band generator is studied in [BKL]. The main property of band generator is that they gives another Garside structure of B_n. That means, almost all results about positive braids B_n^+, such as algorithm to solve word or conjugacy problem remains true with appropriate modification. We do not describe these properties of band generator because we do not use them. See [BKL] for more details. Only we need to prove theorem 1 is estimation of Dehornoy floor using word length of band generators.

Lemma 1. *If a braid $\beta \in B_n$ is conjugate to a braid β' written by a word which consists of products of m band generator ,then*

$$\frac{m}{n} > [\beta]_D$$

holds.

To proof lemma 1, we need property S of Dehornoy ordering, which asserts inserting σ_i strictly increase Dehornoy ordering.

Proposition 2 (Property S of Dehornoy ordering [DDRW])**.** *For any braids $\beta_1, \beta_2 \in B_n$ and $1 \leq i \leq n-1$,*

$$\beta_1 \sigma_i \beta_2 > \beta_1 \beta_2 > \beta_1 \sigma_i^{-1} \beta_2$$

holds.

Proof of lemma 1. We regard braid group B_n as a relative mapping class group of punctured disc $MCG(D_n, \partial D_n)$; That is, a group of isotopy classes of homeomorphisms of D_n whose restriction of ∂D_n are identities. Let $\beta = \alpha^{-1}\beta'\alpha$ and p be a minimal non-negative integer which is larger than $\frac{m}{n}$. We show $\beta < \Delta^{2p}$. The proof of $\Delta^{-2p} < \beta$ is similar. First we consider $m = n$ case.

From property S, it suffices to consider the case $\beta' \in B^{+}_{n\ band}$ because we can delete inverse of band generator by inserting some σ_i without decreasing Dehornoy ordering. Let Γ be a graph in D_n, whose vertices are puncture points of D_n and whose edges are arcs connecting two distinct vertices, lying entirely in upper half of disc. Let D' be a disc which contains whole of Γ and whose boundary belongs to Γ (See figure 1). Let l be a horizontal diameter of D_n which connects all of puncture points, oriented left to right (See also figure 1). From geometric interpretations of Dehornoy ordering given in [FGRRW], $\alpha < \beta$ holds if and only if $\beta(l)$ moves more left than $\alpha(l)$ when two arcs $\beta(l)$ and $\alpha(l)$ are isotoped to have minimum intersections. We denote edge of Γ connecting i-th vertex and j-th vertex by $e_{i,j}$. A band generator $a_{i,j}$ corresponds to a half Dehn-twist along the edge $e_{i,j}$ and a braid $\alpha^{-1}a_{i,j}\alpha$ corresponds to a half Dehn-twist along the arc $\alpha(e_{i,j})$. Similarly, the square of Garside fundamental braid Δ^2 corresponds to Dehn-twist along $\partial D'$. Since $\partial D'$ consists of n edges, so does $\partial\alpha(D')$. Take edges $e'_1, e'_2 \cdots, e'_n$ of $\alpha(\Gamma)$ so that edge-path $e'_1 \cup e'_2 \cdots \cup e'_n$ forms $\partial(\alpha(D'))$. We choose e'_1 be a edge which intersect the arc l at leftmost points: That means, the first intersection point of $\alpha(\Gamma)$ with l is an intersection point of e'_1 with l. Then the maximal element of n half Dehn-twist along $\alpha(\Gamma_{i,j})$ is $b_1b_2b_3\cdots b_n$ where b_i is a half Dehn-twist along the edge e'_i because all of the other braid words change l at further points from starting point of l than $b_1b_2b_3\cdots b_n$, so $b_1b_2b_3\cdots b_n(l)$ moves "most" left. It is easily checked that the braid $b_1b_2b_3\cdots b_n$ is strictly smaller than Dehn-twist along $\partial\alpha(D')$ namely, Δ^2, therefore we conclude $\beta < \Delta^2$.

Since the braid Δ^2 belongs to the center of B_n, by iterating this argument, we conclude $\beta < \Delta^{2p}$. □

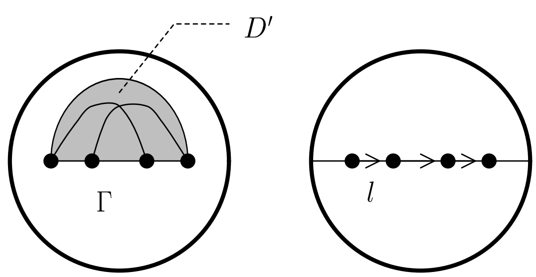

FIGURE 1. Graph Γ

We remark that there exist infinitely many families of left-invariant total orderings of braid groups called *Thurston-type ordering* which contain Dehornoy ordering (See [SW]) as a special one. Thurston type orderings have similar properties of Dehornoy ordering, such as property S. The most of results in the paper also holds if we use Thurston-type ordering instead of Dehornoy ordering after some appropriate modification.

2.2. **Braid foliation.** In this section, we summarize basic machinery of Birman-Menasco's braid foliation theory in case of spanning surface with maximal Euler characteristic, or alternatively, incompressible spanning surface. For details of these techniques and theories, see [BF] or [BM].

Fix an unknot $A \in S^3$, called *axis* and choose a meridinal disc fibration $H = \{\ H_\theta | \theta \in [0, 2\pi]\}$ of the solid torus $S^3 \backslash A$. An oriented link L in $S^3 \backslash A$ is called *closed braid* with axis A if L intersects every fiber H_θ transversely and each fiber is oriented so that all intersections of L are positive. It is easy to see a closed braid L intersects every fiber H_θ in the same number of points, and we call this number *braid index* of L. Note that closed braids $\widehat{\beta}$ obtained by braids $\beta \in B_n$ as usual way are indeed a closed braid with $z-$axis and braid index is n. Conversely, if we cut solid torus $S^3 \backslash A$ along the fiber H_0, we obtain a braid β. It is known that isotopy in $S^3 \backslash A$ does not change conjugacy class of β and if the isotopy fixes $H_0 \cap \widehat{\beta}$, obtained braid β does not change as a element of braid group though geometrical configuration of strands are varied.

Let F be an orientable, connected spanning surface of L with maximal Euler characteristics. An orientation of F is defined so that $L = \partial F$ holds. We remark that such a surface is always incompressible in $S^3 \backslash L$. Then the intersections of fiber $\{H_\theta\}$ with F induce a singular foliation of F. The leaves of this foliation are connected components of intersection with fibers. Braid foliation techniques are, in short, modifying this foliation simpler as possible and obtain standard position or representation of braids. By the argument in [BF], F can be isotoped to "general" position with respect to the fibration which satisfies

- **(1):** Axis A pierces F transversely in finitely many points.
- **(2):** For each point $v \in A \cap F$, there exists neighborhood N_v of v such that $F \cap N_v$ is radially foliated disc.
- **(3):** All but finitely many fibers H_θ intersects F transversely, and each of the exceptional fiber is tangent to F at exactly one point. Moreover, each point of tangency is saddle tangency and lies in the interior of $F \cap H_\theta$.

Notice that the condition above is a bit stronger than usual general position arguments, since usual general position arguments only tell us that each tangency is local minimum, maximum, or saddle. This strong sense of general position is achieved by first putting F in usual general position, and then deleting local minimum or maximum tangencies.

We say fiber is H_θ *regular* if H_θ transverse F and *singular* if H_θ tangent to F. In the foliation of F, we can assume there are only two types of non-singular leaves. First one is *a-arc*, which is a arc with one boundary point on L and the other on A. The other one is *b-arc*, which is a arc with both endpoints on A. We can delete leaves which are simple closed curves. We say b-arc b is essential if both components of $H_\theta \backslash b$ are pierced by L.

As shown in [BF], we can assume one more condition about foliation of F.

(4): every b-arc is essential.

This condition is achieved by deleting all inessential leaves by pushing across axis. From now on, we always assume that F satisfies condition (1)-(4).

We call an intersection point of A with F *vertex*. Each vertex p, the valance of vertex p is, by definition, the number of singular leaves which pass p. We say singular point is *aa-singularity* if the singular point is derived from two a-arcs. *ab-singularity* and *bb-singularity* are defined by the same way. Each type of singularity has neighborhood shown in figure 2, and we say such neighborhood *regions*. The decomposition into regions defines cellular decomposition of F. We call this cellular decomposition *tiling*. It is directly checked that the valance of vertex previously defined one, the number of singular leaves passing the vertex, coincide with the usual meaning of valance in this cellular decomposition.

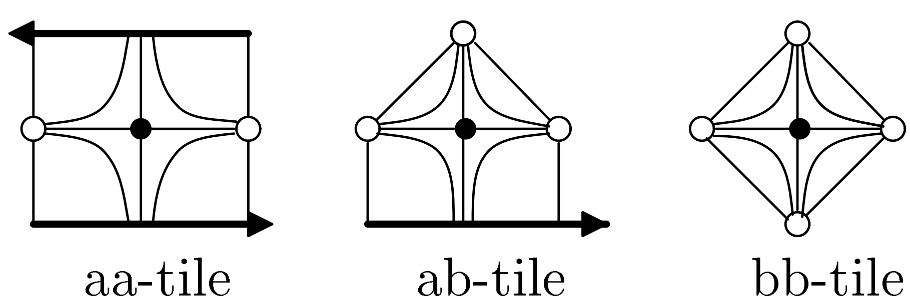

FIGURE 2. aa-,ab-,bb- tiles

3. Proof of theorem 1

In this section we prove theorem 1. The strategy is essentially the same as that of our previous paper [I]. We first establish Euler characteristic formula for spanning surface, then estimate Dehornoy floor using valance of vertices in the tiling of the surface. Combining these two results, we obtain desired estimation.

Let F be a spanning surface of a closed $n-$ braid $\widehat{\beta}$ with maximal Euler characteristics. Let $V(a,b)$ be the number of vertices in the tiling of F whose valance is $a+b$ and having a a-arcs as edges and b b-arcs as edges. We call such a vertex *type*(a,b)*-vertex*. The following lemma has proved in [BM] using Euler characteristic arguments of cellular decomposition determined by decomposition of tiles.

Lemma 2 (Birman-Menasco([BM]))**.**

$$\begin{aligned} & 2V(1,0)+2V(0,2)+V(0,3)-4\chi(F) \\ = \quad & V(2,1)+2V(3,0)+\sum_{v=4}^{\infty}\sum_{\alpha=0}^{v}(v+\alpha-4)V(v,\alpha) \end{aligned}$$

Now we establish an estimation of Dehornoy floor using valance of vertex in tiling of F. The argument is very similar to one appeared in [I], but since there are two types of edges we require some additional arguments.

Lemma 3. *If F has (a,b)-type vertices, $[\beta]_D < a+\frac{b}{2}$. Especially, if both a and b are non-zero, then $[\beta]_D < a+\frac{b}{2}-\frac{1}{2}$.*

Proof. Let v be a (a,b)-type vertex of F and $\{H_{\theta_i} \mid i=1,2,\cdots,a+b \;\; \theta_i < \theta_{i+1}\}$ be a sequence of singular fibers containing singular leaves which pass vertex v. We denote a leaf in H_θ which passes v by δ_θ . The first task we have to do is that to modify closed braid $\widehat{\beta}$ to special position so that we can obtain the description of the braid. Take a sufficiently small $\varepsilon > 0$ so that there are no singularities in the interval $[\theta_i-\varepsilon, \theta_i+\varepsilon]$ except H_{θ_i}. We modify closed braid $\widehat{\beta}=L$ so that in each intersection of F with fiber satisfies following condition.

1. $H_{\theta_i \pm \varepsilon} \cap L$ consists of the same n points in $H_{\theta_i \pm\varepsilon} \cong D^2$ which lies on horizontal diameter.
2. Each vertices and a-arcs in a fiber $H_{\theta_i\pm\varepsilon}$ lies in lower half of disc
3. The vertex v lies at leftmost position in the boundary of lower half of disc.
4. If all of $\{l_\theta\}$ are b-arc in the interval $[\theta_i+\varepsilon, \theta_{i+1}-\varepsilon]$, then these b-arcs do not move in $[\theta_i+\varepsilon, \theta_{i+1}-\varepsilon]$.

From condition (1), we can always consider sub-braiding in each interval $[\theta_i - \varepsilon, \theta_i + \varepsilon]$ and $[\theta_i + \varepsilon, \theta_{i+1} - \varepsilon]$. We denote this modified braid by β'. Since above modification is merely isotopy of closed braid and surface in the complement of axis, so β' are conjugate to β. We remark that during the isotopy, $H_0 \cap \widehat{\beta}$ can be change to make condition (3) above is satisfied so in general β' is not identical with β.

First of all, we study braiding in $[\theta_i - \varepsilon, \theta_i + \varepsilon]$. It is directly checked by seeing corresponding moves of leaves and braid strands near the singularity, braiding in $[\theta_i - \varepsilon, \theta_i + \varepsilon]$ can be written by

$$a_{1,j} = (\sigma_j \sigma_{j-1} \cdots \sigma_2 \sigma_1^{\pm 1} \sigma_2^{-1} \cdots \sigma_j^{-1})$$

, which is a merely band generator and corresponds to adding a twisting band, if singularity contained in H_{θ_i} is an aa-singularity. If singularity in H_{θ_i} is an ab-singularity, then braiding in $[\theta_i - \varepsilon, \theta_i + \varepsilon]$ is given by

$$(\sigma_j \sigma_{j-1} \cdots \sigma_1) \text{ or } (\sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_j^{-1}).$$

See [BH] for detailed arguments to obtain braid word via braid foliation. We say ab-singularity giving former types of words *type a-b* because along the neighborhood of such ab-singularity leaf l_θ is changed a-arc to b-arc. Similarly, ab-singularity giving latter type of words *type b-a*. See figure 3 and 4. We remark that each type of ab-singularity, there exists two kinds of ab-singularity. The first kind is that a-arc is attached to b-arc from right, which is depicted in the figure 4(a) and the other is from left, as depicted in the figure 4(b). We remark that though both kinds of ab-singularity gives the same braid words, configuration of b-arcs and strands after singularity are different. In the remaining case, when singularity contained in H_{θ_i} is a bb-singularity, we can assume in the interval $[\theta_i - \varepsilon, \theta_i + \varepsilon]$, L is not braided because all of changes of leaves occur in b-arc so a-arcs do not move.

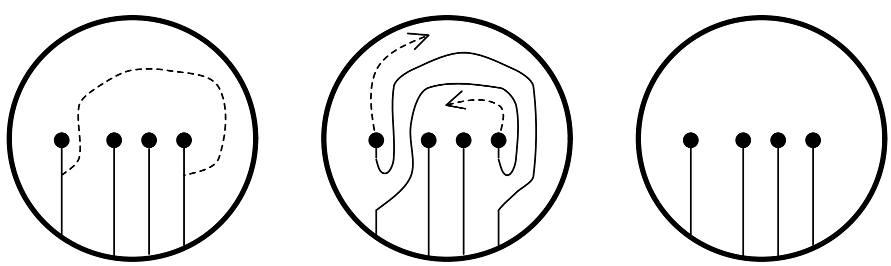

FIGURE 3. Corresponding moves of leaves and strands in aa-singularity

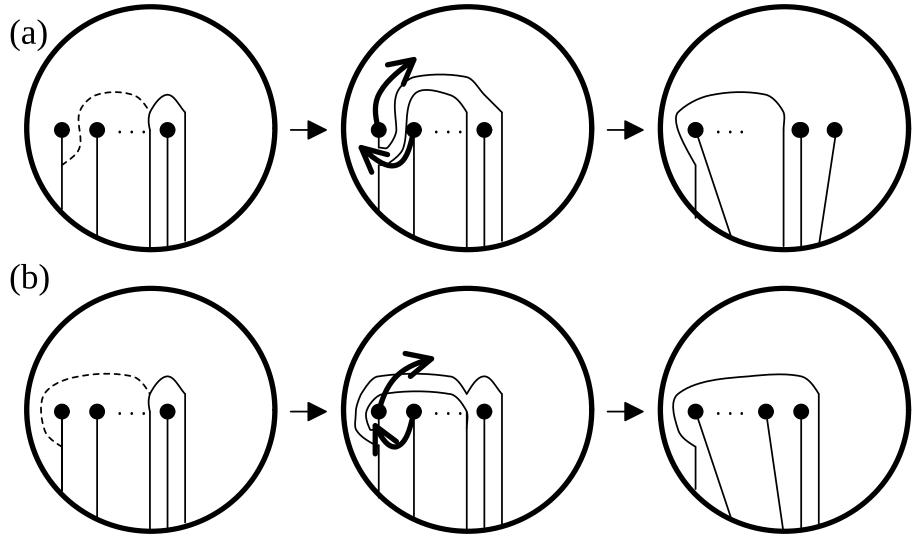

FIGURE 4. Corresponding moves of leaves and strands in ab-singularity

Next we study a braiding in the interval $I = [\theta_i + \varepsilon, \theta_{i+1} - \varepsilon]$.

There exist two types of such intervals $[\theta_i + \varepsilon, \theta_{i+1} - \varepsilon]$: Namely,

(1) An interval between two aa-singularities, or interval between aa-singularity and ab-singularity, and

(2) An interval between ab-singularity and aa-singularity, or an interval between two bb-singularities.

We call each type of intervals *type A, type B* respectively because in type A (resp. type B) interval, leaf l_θ is a-arc (resp. b-arc).

In a type A interval I, l_θ is always a-arc. Let δ be a strand of link L which corresponds to boundary point of l_θ. In braid diagram, δ corresponds to 1-st strands. Then, δ is never braided with other strands because in the interval I, l_θ do not move. Therefore we can write braiding in I as left figure of figure 5.

In the case of type B interval I, since δ_θ is essential, the leaf δ_θ separates fiber H_θ into two components, both of which are pierced by L. That means we can write braiding in I as a right figure of figure 5.

Now, we can obtain whole braiding of L. A next step is to simplify obtained braid so that it contains less $\sigma_1^{\pm 1}$.

First we consider a braiding in the interval J, which is an union of ε-neighborhood of ab-singular point N and an adjacent type B interval I. Our previous argument shows braiding in J can be written as in the right figure of figure 6. It is directly checked that braiding in J are

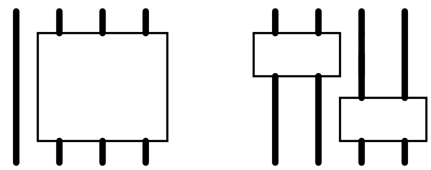

Figure 5. Braiding in intervals

modified as shown in figure 6 according to kinds of ab-singularity (we show type a-b case in the figure. The case of type b-a is similar.) If ab-singularity is from right, then braid box in I which contains strands 1 can be shifted to across braiding in N so that in the interval J braiding contains only one $\sigma_1^{\pm 1}$. If ab-singularity is from left, then braiding in N is amalgamated into braid box in I which contains strands 1 so that we can neglect braiding derived from ε-neighborhood of ab-singular point.

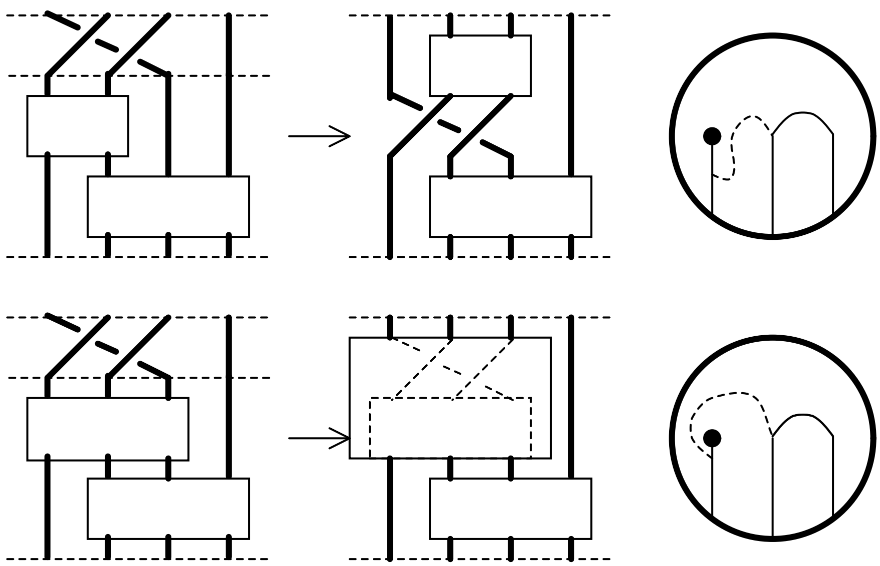

Figure 6. Modification near the ab-singularity

Next we observe that if two type B intervals I_1, I_2 are adjacent, then by exchanging an order of braid box in I_2, we can modify braiding in the interval $I_1 \cup I_2$ as in figure 7 which contains only one σ_1 and σ_1^{-1}. Such an exchange of braid blocks is indeed possible because two braid blocks are separated by b-arc l_θ, so they are non braided each other.

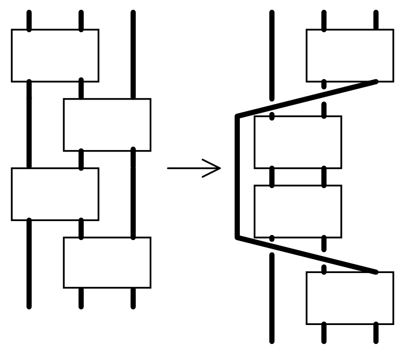

FIGURE 7. Modification in union of two type BB intervals

Now we estimate how many σ_1 and σ_1^{-1} the braid β' can contain. If $b = 0$, then from argument above, braid β' has at most a $\sigma_1^{\pm 1}$ and if $a = 0$, then braid β' has at most $\frac{b}{2}$ $\sigma_1^{\pm 1}$. Therefore in these cases, we obtain desired estimation by proposition 1. So now, we assume that $a \neq 0$ and $b \neq 0$. In this case, there must exist at least two ab-singularities and the cycle of singularities around v are decomposed to some repetition of sub-cycles

$$\{\underbrace{aa \to aa \to \cdots \to aa}_{k\text{times}} \to ab \to \underbrace{bb \to bb \to \cdots \to bb}_{l\text{times}} \to ab\}$$

which contains k aa-singularity and l bb-singularity (k, l might be zero).

From above arguments, one aa-singularity gives at most one σ_1 or $\sigma_1 \pm 1$ and type A intervals and bb-singularities have no contribution to the number of $\sigma_1^{\pm 1}$. Now we reduce the number of σ_1, σ_1^{-1} derived from ab-singularities and type B intervals using modification described above.

If two ab-singularity is from left, then we can amalgamate braiding derived from ab-singularity into adjacent braid blocks derived from type B interval. In this case, we only consider the contribution of $\sigma_1^{\pm 1}$ derived from type B intervals. In this case, there exist $l + 1$ type B intervals, so these intervals contribute at most $\frac{l+1}{2}$ $\sigma_1^{\pm 1}$. Thus, in this case the sub-cycle contains at most $\frac{l+1}{2} + k$ σ_1 or σ_1^{-1}.

Now assume that one of ab-singularity is from left and the other is from right. Then the ab-singularity from right is modified together with adjacent type B interval, and gives one σ_1 or σ_{-1}. This modification delete one type B intervals. The ab-singularity from left is amalagamated with adjacent type B intervals, so we can neglect it. The number of remaining type B intervals is l, so type B intervals give at most $\frac{l}{2}$ $\sigma_1^{\pm 1}$. As a result, in this case there are at most $\frac{l}{2}+1+k$ σ_1 or σ_1^{-1} in the sub-cycle.

Finally, if both of ab-singularity is from right, then ab-singularities, modified with adjacent type B intervals, gives one σ_1 and σ_1^{-1}. This modification delete two type B intervals. The number of remaining type B intervals is $l-1$, and they contribute at most $\frac{l-1}{2}$ $\sigma_1^{\pm 1}$. Therefore in this case sub-cycle contains at most $\frac{l-1}{2}+1+k$ $\sigma_1^{\pm 1}$.

From above argument, we conclude that such sub cycle contains at most $k+1+\frac{l}{2}$ $\sigma_1^{\pm 1}$. Therefore if we want to braid β' contain σ_1 or σ_1^{-1} as many as possible, then cycle of singularity around v must contain only one sub-cycle described above. In such cycle, there are $a-1$ aa-singularities and $b-1$ bb-singularities, so we conclude that modified braid β' has word representation which has at most $a+\frac{b-1}{2}$ σ_1 or σ_1^{-1}. Since original braid β is conjugate to β', by proposition 1 we obtain $[\beta]_D < a+\frac{b}{2}-\frac{1}{2}$. □

Now we are ready to prove theorem 1.

Proof of theorem 1. Let L be an oriented link and F be a Seifert surface of L with maximal Euler characteristics. Take a closed braid representative $\widehat{\beta}$ of L and isotope F so that F is braid-foliated position. If there exists at least one vertices of either type (2,0), (1,1), (1,2), (0,2), (0,3) or (0,4), then lemma 3 shows $[\beta]_D < 2$. Therefore we can assume that there exist no vertices of such types. Thus, now Euler characteristic formula is

$$-4\chi(F) = V(2,1) + 2V(3,0) + \sum_{v=4}^{\infty}\sum_{a=0}^{v}(v+a-4)V(a,v-a)$$

.

First assume that F is foliated by only a-arcs. In this case, there exist exactly n vertices on F and exactly$-\chi(F)+n$ aa-singularity on F because aa-singularity can be seen as a twisted band attached to discs which are neighborhood of vertices ([BH]). Therefore, a braid β' is written as a product of $n-\chi(F)$ band generators, so lemma 1 shows $[\beta']_D < 1-\frac{\chi(F)}{n}$. Since original braid β is conjugate to β', we establish $[\beta]_D < 1-\frac{\chi(F)}{n} < 2-\frac{2\chi(F)}{n+2}$ from lemma 1.

Thus, we can assume F contains both b-arc and a-arcs. In such cases there exist at least $n+2$ vertices in the foliation, so for any $V(a,b)$ which appears in right-side of Euler characteristics formula and minimize $a+\frac{b}{2}=\frac{v+a}{2}$,

$$-4\chi(F) \geq (2a+b-4)(n+2)$$

holds. Therefore if $V(a,b)$ is non-zero and minimize $a+\frac{b}{2}$, then inequality

$$\frac{-2\chi(F)}{n+2}+2 \geq a+\frac{b}{2}$$

holds. Therefore lemma 3 gives desired estimation. □

Now we state a corollary of theorem 1 concerning the author's previous paper. In the author's previous paper [I], the author obtained following constructions of hyperbolic knots. Let $MCG(D_n)$ be a mapping class groups of n-punctured disc and $\pi : B_n \rightarrow MCG(D_n)$ be a natural projection regarding B_n as isotopy classes of homeomorphisms of disc. For a pseudo-Anosov mapping class $[f] \in MCG(D_n)$, let $P([f]) = \{\widehat{\beta} \mid \beta \in \pi^{-1}([f]), [\beta]_D \geq 3\}$. In [I], the author shows $P([f])$ consists of infinite number of hyperbolic knots and for another pseudo-Anosov mapping class $[g]$ which is not conjugate to $[f]$, the intersection of $P([f])$ and $P([g])$ is finite.

Since every hyperbolic knots are represented by a closure of pseudo-Anosov braid, one might expect this construction of hyperbolic knots produces *all* hyperbolic knots. However, this is not true.

Corollary 2. *Genus one hyperbolic knots do not appear as an element of $P([f])$.*

Proof. Let K be a genus one hyperbolic knot. By corollary 1, we obtain for every closed braid representative $\widehat{\beta}$ of K, $[\beta]_D < 3$. Thus, such a knot do not appear as a element of $P([f])$. □

Finally, we mention a conjecture related to main result of the paper. Though corollary 1 concerns about 3-genus of knots and Dehornoy floor, the author (weakly) conjectures that the same inequality holds also between slice genus of knots and Dehornoy floor of braids. In some cases this conjecture is easily confirmed: Closed braid representative of a slice knot $K\sharp\overline{K}$ has Dehornoy floor at most 1 so conjecture is trivially holds and slice-Bennequin inequality implies quasi-positive braid also satisfies this conjecture.

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