# $3k-4$ THEOREM FOR ORDERED GROUPS

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Abstract. Recently, G. A. Freiman, M. Herzog, P. Longobardi, M. Maj proved two 'structure theorems' for ordered groups [1]. We give elementary proof of these two theorems.

## 1. Introduction

For any group $G$ (written multiplicatively) and a subset $S$ of $G$ we define $S^2 = \{ab : a, b \in S\}$. Then, the main theorem of [1] is

**Theorem 1.1.** *[Theorem 1.3, [1]] Let $G$ be an ordered group and $S$ be a finite subset of $G$. If $|S^2| \leq 3|S| - 3$ then the subgroup generated by $S$ is an abelian subgroup of $G$.*

As a corollary to Theorem 1.1, they deduce a $3k - 4$ type theorem for ordered groups.

**Theorem 1.2.** *[Corollary 1.4, [1]] Let $G$ be an ordered group and $S$ be a finite subset of $G$ with $|S| = k \geq 3$. If $|S^2| \leq 3|S| - 4$, then there exist two commuting elements $x, y$ such that $S \subset \{yx^i : 0 \leq i \leq N\}$ for $N = |S^2| - |S|$.*

We give elementary proofs of Theorem 1.1 and Theorem 1.2. We shall always assume that $G$ is an ordered group and $S$ is a finite subset of $G$ with $k$ elements. We shall write $S = \{x_1, \ldots, x_k\}$ and assume that $x_1 < \ldots < x_k$.

## 2. Proofs

As in the case of integers, the following inequality holds:

$$|S^2| \geq 2|S| - 1. \tag{1}$$

In equation (1) equality holds only if $S$ is a geometric progression $\{yx^i : 0 \leq i \leq k\}$, with $x, y$ two commuting elements of $G$.

**Lemma 1.** *If $S$ is not a geometric progression then $|S^2| \geq 2|S|$.*

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*Proof.* Let $S = \{x_1 < \ldots < x_k\}$. Certainly

$$x_1x_1 < x_1x_2 < \ldots < x_1x_k < x_2x_k < \ldots < x_kx_k$$

are $2|S| - 1$ distinct elements in $S^2$. If $|S^2| < 2|S|$ then

$$\{x_1x_1, x_1x_2, \ldots, x_1x_k, x_2x_k, \ldots, x_kx_k\} = S^2.$$

Now, consider the elements $x_2x_1 < x_2x_2 < \ldots < x_2x_k$. All these elements are in $S^2$ and $x_1x_1 < x_2x_1, \ldots, x_2x_{k-1} < x_2x_k$. Thus we must have

$$x_2x_1 = x_1x_2, x_2x_2 = x_1x_3, x_2x_3 = x_1x_4, \ldots, x_2x_{k-1} = x_1x_k.$$

From the above relations it follows that $x_1$ and $x_2$ commute and for $i > 2$, $x_i$ is contained in the subgroup generated by $x_1, \ldots, x_{i-1}$. Consequently we get that each $x_i$ commutes with each $x_j$ for $i, j = 1, \ldots, k$.

Put $y = x_1, x = x_2x_1^{-1}$, then $x$ and $y$ commute and $S = \{y, xy, x^2y, \ldots, x^{k-1}y\}$. Thus, if $S$ is not a geometric progression then $|S^2| \geq 2|S|$. □

*Proof of Theorem 1.2.* We shall use induction on $k$. For $k = 3$, we have $|S^2| \leq 5$. We have five distinct elements $x_1^2 < x_1x_2 < x_2^2 < x_2x_3 < x_3^2$ in $S^2$. Since $x_1x_3 \in S^2$, so $x_1x_3$ must equal to one of these five elements. Using the order relation, we get $x_1x_3 = x_2^2$. Similarly, we get $x_1x_2 = x_2x_1$. Let $y = x_1$ and $x = x_2x_1^{-1}$. Then $x$ and $y$ commute and $S = \{y, yx, yx^2\}$.
Now we assume that $k \geq 4$ and the theorem is true for any subset $T$ of $G$ with $|T| \leq k - 1$. Put $T = \{x_1, \ldots, x_{k-1}\}$.
Case (1): $|T^2| \leq 3|T| - 4$.
By induction hypothesis, there are commuting elements $x, y$ such that $T \subset \{yx^j : j = 0, \ldots, M\}$ with $M = |T^2| - |T|$.
In case $x_kT \cap T^2 = \emptyset$, then, taking $x_k^2$ in account, we see that $|S^2| \geq |T^2| + (|T| + 1)$. Since $|T^2| \geq 2|T| - 1$, we immediately obtain $|S^2| \geq 3|S| - 3$, which contradicts the hypothesis. Thus, we get $x_kT \cap T^2 \neq \emptyset$. Consequently, there are $yx^i, yx^u, yx^v \in T$ such that $x_kyx^i = yx^uyx^v$. This gives $x_k = yx^{(u+v-i)}$ and $S \subset \{yx^j : j = 0, \ldots, M'\}$ with $M' = \max\{M, u + v - i\}$. Clearly the map $yx^j \mapsto j$ gives a $2-isomorphism$ of $S$ with a subset of $\mathbb{Z}$. From the Freiman's $3k - 4$-theorem for integers, it follows that $M' \leq N$, and the theorem is proved.
Case (2): $|T^2| \geq 3|T| - 3 = 3|S| - 6$. Using the order relation of $G$ we see that the elements $x_k^2$ and $x_kx_{k-1}$ of $S^2$ are not in $T^2$. Consider the element $x_{k-1}x_k$ of $S^2$. If $x_{k-1}x_k \neq x_kx_{k-1}$ then we get $|S^2| \geq |T^2| + 3$, which contradicts the hypothesis. So, we obtain $x_{k-1}x_k = x_kx_{k-1}$. Next, we consider the element $x_{k-2}x_k$ of $S^2$. If $x_{k-2}x_k \neq x_{k-1}^2$, then we already get $|S^2| \geq |T^2| + 3$, leading to a contradiction. Similarly it follows that $x_kx_{k-2} = x_{k-1}^2$. Thus we have

$$x_{k-1}x_k = x_kx_{k-1}, x_{k-2}x_k = x_kx_{k-2} = x_{k-1}^2.$$

Put $y = x_k, x = x_{k-1}x_k^{-1}$. Then $x$ and $y$ commute and $x_k = y, x_{k-1} = yx, x_{k-2} = yx^2$. Considering the elements $x_{k-3}x_k, x_{k-4}x_k, \ldots, x_1x_k$ successively we see that each of

$x_i$ is of the form $yx^{t_i}$. Clearly $S$ is $2-$*isomorphic* to the subset $\{t_i : 1 \leq i \leq k\}$ of $\mathbb{Z}$. Now the theorem follows from the Freiman's $3k-4$-theorem for integers. $\square$

*Proof of Theorem 1.1.* We shall use induction on $k$. For $k = 1, 2$, the theorem holds trivially. Now, let $k \geq 3$ and assume that the theorem is true for any set $T$ with $|T| \leq k-1$. Put $T = \{x_1, \ldots, x_{k-1}\}$.
Case (1): $|T^2| \leq 3|T| - 3$.
By induction hypothesis, $T$ generates a commutative subgroup. If $x_kT \cap T^2 \neq \emptyset$ or $Tx_k \cap T^2 \neq \emptyset$ then $x_k$ lies in the subgroup generated by $T$. Consequently, $S$ generates a commutative subgroup. So we assume that $x_kT \cap T^2 = \emptyset$ and $Tx_k \cap T^2 = \emptyset$. Using the order relation, we see that $x_k^2 \notin T^2 \cup x_kT$, so we obtain

$$|S^2| \geq |T^2| + |T| + 1. \tag{2}$$

If $T$ is not a geometric progression then, using Lemma 1 in (2), we see that $|S^2| \geq 3|S| - 2$, which contradicts the hypothesis. Thus, $T$ must be a geometric progression. Next, observe that if $x_kT \neq Tx_k$ then we have an element in $Tx_k$ which is not in $T^2 \cup x_kT \cup \{x_k^2\}$. This leads to

$$|S^2| \geq |T^2| + |T| + 1 + 1. \tag{3}$$

From this one obtains $|S^2| \geq 3|S| - 2$, which contradicts the hypothesis. Thus, we must have $x_kT = Tx_k$. Now using the order relation we see that $x_k$ commutes with all the elements of $T$ and consequently $S$ generates an abelian group.
Case (2): $|T^2| > 3|T| - 3$.
As in the proof of Theorem 1.2 (the arguments used in Case (2)) we see that either $|S^2| \geq |T^2| + 3$ or $S = \{yx^{t_i} : 1 \leq i \leq k\}$ with commuting elements $x$ and $y$. The former leads to a contradiction and hence we get $S = \{yx^{t_i} : 1 \leq i \leq k\}$ with commuting elements $x$ and $y$. This proves the theorem. $\square$

**Remark 1.** *From the proof of Theorem 1.2 it is clear that the subgroup generated by $S$ (with $|S| > 2$) is, in fact, generated by $|S| - 1$ or less elements.*

## References

[1] G. A. Freiman, M. Herzog, P. Longobardi, M. Maj, *Small doubling in ordered groups* J. Aust. Math. Soc. **96** (2014), no. 3, 316-325.

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