# On the free product of ordered groups

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One of the fundamental questions of the theory of ordered groups is what abstract groups are orderable. E. P. Shimbireva [2] showed that a free group on any set of generators can be ordered. This leads to the following problem: under what conditions is it possible to order a free product of arbitrary groups?

Using the matrix presentation method for groups proposed by Malcev [1], in the present work we establish the orderability of a free product of arbitrary ordered groups.

**Definition 1.** An *ordered group* is a group endowed with a relation $>$, satisfying the following conditions:

1. For any elements $x$ and $y$ of the group either $x > y$, or $y > x$, or $x = y$.
2. If $x > y$ and $y > z$, then $x > z$.
3. If $x > y$, then $axb > ayb$ for any elements $a$ and $b$ of the group.

**Definition 2.** An *ordered ring (field)* is a ring (field) such that:

1. the additive group of the ring (field) is ordered, and
2. for any elements $a$, $x$, $y$ of the ring (field),

$$(a > 0 \text{ and } x > y) \quad \Longrightarrow \quad (ax > ay \text{ and } xa > ya).$$

**Definition 3.** The *group algebra* $\Bbbk\mathfrak{G}$ of a group $\mathfrak{G}$ over a field $\Bbbk$ is the algebra whose elements are formal finite linear combinations of elements of $\mathfrak{G}$ with coefficients in $\Bbbk$. These sums are multiplied and added in the usual way. A group algebra has the obvious unit $1e$, where $e$ is the identity element of $\mathfrak{G}$ and 1 the unit of $\Bbbk$.

**Lemma 1.** *If $\Bbbk$ is an ordered field and $\mathfrak{G}$ an ordered group, then $\Bbbk\mathfrak{G}$ is orderable.*

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*Proof.* Let $A$ and $A'$ be elements of $\Bbbk\mathfrak{G}$ under the conditions of the lemma. Then they can be written as

$$A = \sum_{i=1}^{n} \alpha_i a_i, \qquad A' = \sum_{i=1}^{n} \alpha_i' a_i,$$

where some of the $\alpha_i$ and $\alpha_i'$ might be zero, and $a_1 > \ldots > a_n$. We set $A > A'$ if for some $r \in \{1, \ldots, n\}$,

$$\alpha_1 = \alpha_1', \quad \ldots, \quad \alpha_{r-1} = \alpha_{r-1}', \quad \alpha_r > \alpha_r'.$$

It is easy to check that the conditions from Definition 2 hold. □

We call a *triangular matrix* any matrix, finite or infinite, with zeroes under the main diagonal.

**Lemma 2.** *The set of all triangular matrices with entries in an ordered unital ring, and with every element on the main diagonal positive and invertible, is an orderable group.*

*Proof.* Triangular matrices of the form described in the statement clearly form a group. Let $X$ and $Y$ be such matrices. We will call *preceding entries* to a given entry $x_{ik}$, those[1] $x_{nm}$ located to the right of or on the main diagonal, for which

$$\begin{array}{lll} n - m \leq k - i & \text{when} & m < i, \quad \text{and} \\ n - m < k - i & \text{when} & m \geq i. \end{array}$$

Say that $X > Y$ if either of the following conditions holds:

- $x_{ii} = y_{ii}$ for $i = 1, \ldots, k-1$, and $x_{kk} > y_{kk}$ for some $k$,
- $x_{ik} > y_{ik}$ for some $k > i$, and their preceding entries coincide.

One easily checks that the conditions of Definition 1 are satisfied. □

**Lemma 3.** *The direct product of two ordered groups is orderable.*

*Proof.* Let $\mathfrak{A}$ and $\mathfrak{B}$ be ordered groups. Say that $(a, b) > (a', b')$ in $\mathfrak{A} \times \mathfrak{B}$ if either $a > a'$, or $a = a'$ and $b > b'$. It is easy to check that the conditions from Definition 1 hold. □

We denote by $\mathfrak{M}$ the direct product of two ordered groups $\mathfrak{A}$ and $\mathfrak{B}$. A pair of the form $(a, e_1)$ where $e_1$ is the identity of $\mathfrak{B}$ will be denoted simply by $a$, and a pair of the form $(e, b)$ where $e$ is the identity of $\mathfrak{A}$ will be denoted by $b$.

[1] Translators' note: we believe that there is a mistake here, $x_{nm}$ should probably be replaced with $x_{mn}$.

Consider now the following transcendental triangular matrix:

$$X = \begin{Vmatrix} 1 & x_{12} & x_{13} & x_{14} & \cdot & \cdot & \cdot \\ & 1 & x_{23} & x_{24} & \cdot & \cdot & \cdot \\ & & 1 & x_{34} & \cdot & \cdot & \cdot \\ & & & 1 & \cdot & \cdot & \cdot \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{Vmatrix}$$

We denote by $\mathfrak{G}$ the free abelian group generated by the entries $x_{ij}$ of $X$. This group is orderable (see [2] and references therein). By Lemma 1, the group algebra $\mathfrak{K} = \mathbb{Q}\mathfrak{G}$ is orderable, and thus has no zero divisors. The field of fractions $\mathrm{Frac}(\mathfrak{K})$ of this algebra is also orderable [3]. Consider the group algebra $\mathfrak{L} = \mathrm{Frac}(\mathfrak{K})\mathfrak{M}$, where $\mathfrak{M} = \mathfrak{A} \times \mathfrak{B}$ as above. According to Lemmas 1 and 3, the algebra $\mathfrak{L}$ is orderable.

**Lemma 4.** *Consider the diagonal matrix*

$$A = \begin{Vmatrix} 1 & & & & & & \\ & a & & & & & \\ & & 1 & & & & \\ & & & a & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{Vmatrix}$$

*where* $1$ *is the unit of* $\mathfrak{L}$ *and* $a \in \mathfrak{L}$ *is neither* $0$ *nor* $1$. *Then every entry of the matrix* $B = X^{-1}AX$ *located to the right of or on the main diagonal is non-zero.*

*Proof.* Put $X^{-1} = (y_{ik})$ and $B = (b_{ik})$. Clearly[2],

$$y_{in} = -x_{in} + \sum_{i<\alpha_1<n} x_{i\alpha_1}x_{\alpha_1 n} - \sum_{i<\alpha_1<\alpha_2<n} x_{i\alpha_1}x_{\alpha_1\alpha_2}x_{\alpha_2 n} + \cdots + \\ +(-1)^{n-i}x_{i,\,i+1}x_{i+1,\,i+2}\ldots x_{n-1,\,n}$$

[2] Translators' note: we corrected the last term of the formula given for $y_{in}$. Note also that this formula holds only for $i \neq n$, as $y_{ii} = 1$. As a result, the very last formula of this proof is slightly incorrect when $i$ is odd, but the main point—that the coefficient of $b_{ik}$ is not 0—seems to hold true after all.

and

$$b_{ik} = 1(y_{i1}x_{1k} + y_{i3}x_{3k} + \cdots + y_{i,\,2l+1}x_{2l+1,\,k}) + \\ a(y_{i2}x_{2k} + y_{i4}x_{4k} + \cdots + y_{i,\,2r}x_{2r,\,k}).$$

From this follows:

$$y_{i1}x_{1k} + y_{i3}x_{3k} + \cdots + y_{i,\,2l+1}x_{2l+1,\,k} \; = \\ -\sum x_{in}x_{nk} + \sum \sum_{i<\alpha_1<n} x_{i\alpha_1}x_{\alpha_1 n}x_{nk} - \sum \sum_{i<\alpha_1<\alpha_2<n} x_{i\alpha_1}x_{\alpha_1\alpha_2}x_{\alpha_2 n}x_{nk} + \cdots \,,$$

where the external sums are over all odd integers $n$ between $i$ and $k$. This equality shows that the coefficient of 1 in $b_{ik}$ is non-zero, and so $b_{ik} \neq 0$. □

**Theorem.** *The free product of two ordered groups can be endowed with a group order whose restriction to each factor is the original order.*

*Proof.* Consider, together with the triangular matrix $X$ introduced before, the following transcendental triangular matrices:

$$Y = \left\| \begin{array}{cccccc} 1 & y_{12} & y_{13} & y_{14} & \cdot & \cdot & \cdot \\ & 1 & y_{23} & y_{24} & \cdot & \cdot & \cdot \\ & & 1 & y_{34} & \cdot & \cdot & \cdot \\ & & & 1 & \cdot & \cdot & \cdot \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{array} \right\|,$$

$$U = \left\| \begin{array}{cccccc} u_1 & & & & & \\ & u_2 & & & & \\ & & u_3 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \end{array} \right\|, \qquad V = \left\| \begin{array}{cccccc} v_1 & & & & & \\ & v_2 & & & & \\ & & v_3 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \end{array} \right\|.$$

Let $\mathfrak{A}$ and $\mathfrak{B}$ be ordered groups. As before, we construct an algebra $\mathfrak{L} = \mathrm{Frac}(\mathbb{Q}\mathfrak{G})\mathfrak{M}$ with $\mathfrak{M} = \mathfrak{A} \times \mathfrak{B}$, where now the free abelian group $\mathfrak{G}$ is generated by the set of all formal entries not only of $X$, but also of $Y$, $U$, and $V$. To every $a = (a, e_1) \in \mathfrak{M}$ we associate the diagonal matrix

$$\overline{A_a} = \left\| \begin{matrix} 1 & & & & & & \\ & a & & & & & \\ & & 1 & & & & \\ & & & a & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{matrix} \right\|,$$

and to every $b = (e, b) \in \mathfrak{M}$ the diagonal matrix

$$\overline{B_b} = \left\| \begin{matrix} 1 & & & & & & \\ & b & & & & & \\ & & 1 & & & & \\ & & & b & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{matrix} \right\|.$$

Clearly the two sets of matrices $\overline{\mathfrak{A}} = \left\{ \overline{A_a} \,\middle|\, a \in \mathfrak{A} \right\}$ and $\overline{\mathfrak{B}} = \left\{ \overline{B_b} \,\middle|\, b \in \mathfrak{B} \right\}$ form groups naturally isomorphic to $\mathfrak{A}$ and $\mathfrak{B}$ respectively.

Put $\overline{\overline{\mathfrak{A}}} = U^{-1}X^{-1}\overline{\mathfrak{A}}XU$ and $\overline{\overline{\mathfrak{B}}} = V^{-1}Y^{-1}\overline{\mathfrak{B}}YV$. We are going to show that the representations of $\mathfrak{A}$ and $\mathfrak{B}$ given by $a \mapsto \overline{\overline{A_a}}$ and $b \mapsto \overline{\overline{B_b}}$ induce a faithful representation of the free product $\mathfrak{A} * \mathfrak{B}$, that is, given elements of $\mathfrak{A} * \mathfrak{B}$ of type

$$r_1 = \prod_1^n a_i b_i, \qquad r_2 = \left( \prod_1^n a_i b_i \right) a_k, \qquad r_3 = b_k \prod_1^n a_i b_i, \qquad r_4 = \prod_1^n b_i a_i,$$

the corresponding matrices

$$R_1 = \prod_1^n \overline{\overline{A_i}}\,\overline{\overline{B_i}}, \quad R_2 = \left( \prod_1^n \overline{\overline{A_i}}\,\overline{\overline{B_i}} \right) \overline{\overline{A_k}}, \quad R_3 = \overline{\overline{B_k}} \prod_1^n \overline{\overline{A_i}}\,\overline{\overline{B_i}}, \quad R_4 = \prod_1^n \overline{\overline{B_i}}\,\overline{\overline{A_i}}$$

are not the identity matrix. We will write down the proof for $R_1$ only, as the three remaining cases are similar.

Every entry $\overline{\overline{a}}_{kl}^{\,i}$ of the matrix $\overline{\overline{A_i}}$ is equal to $u_k^{-1} a'^{\,i}_{kl} u_l$, where $a'^{\,i}_{kl}$ is an entry of $A'_i = X^{-1}\overline{A_i}X$, and $u_k^{-1}$ and $u_l$ are diagonal entries of the matrices $U^{-1}$ and $U$. Similarly, $\overline{\overline{b}}_{kl}^{\,i} = v_k^{-1} b'^{\,i}_{kl} v_l$, where $b'^{\,i}_{kl}$ is an entry of $B'_i = X^{-1}\overline{B_i}X$, and $v_k^{-1}$ and $v_l$ are diagonal entries of the matrices $V^{-1}$ and $V$.

By Lemma 4, every matrix in the groups $\mathfrak{A}' = X^{-1}\overline{\mathfrak{A}}X$ and $\mathfrak{B}' = Y^{-1}\overline{\mathfrak{B}}Y$ different from the identity matrix has only non-zero entries to the right of or on the main diagonal. The entries of the matrix $R_1$ are given by

$$r_{ik} = \sum_{i \leq i_2 \leq i_3 \leq \ldots \leq i_{2n} \leq k} \overline{\overline{a}}^{(1)}_{ii_2} \overline{\overline{b}}^{(1)}_{i_2 i_3} \overline{\overline{a}}^{(2)}_{i_3 i_4} \overline{\overline{b}}^{(2)}_{i_4 i_5} \cdots \overline{\overline{a}}^{(n)}_{i_{2n-1}, i_{2n}} \overline{\overline{b}}^{(n)}_{i_{2n}, k}.$$

Here $i \leq k$. This sum can be regarded as a polynomial in the diagonal entries of $U$, $V$ and of their inverses. The coefficients of this polynomial are products of entries of the matrices $A'_1, B'_1, A'_2, B'_2, \ldots$. Observe that no monomial occurs twice in the sum as it is given. Moreover, every coefficient is non-zero, since it is a product of non-zero elements of the algebra $\mathfrak{L}$, which has no zero divisors.

Therefore, we have a faithful representation of the free product $\mathfrak{A} * \mathfrak{B}$, given by

$$r_i \mapsto R_i.$$

Every diagonal entry of $R_i$ is either the unit of $\mathfrak{L}$ or a positive invertible element of $\mathfrak{L}$ distinct from the unit. It follows then from Lemma 2 that all matrices of all four types $R_i$ together form an orderable group. Therefore, the free product $\mathfrak{A} * \mathfrak{B}$ is orderable. □

The proof presented here for two factors obviously works for any number of factors.

# References


[1] A. Malcev. On isomorphic matrix representations of infinite groups. *Rec. Math. [Mat. Sbornik] N.S.*, 8 (50):405–422, 1940.

[2] H. Shimbireva. On the theory of partially ordered groups. *Rec. Math. [Mat. Sbornik] N.S.*, 20(62):145–178, 1947.

[3] B. L. van der Waerden. *Modern Algebra. Vol. I.* M.–L., 1934.