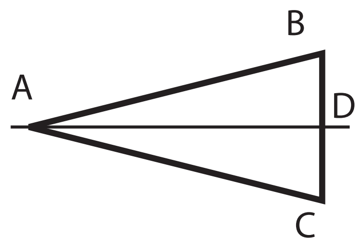
George Polya: "Geometry is the science of correct reasoning using incorrect figures."

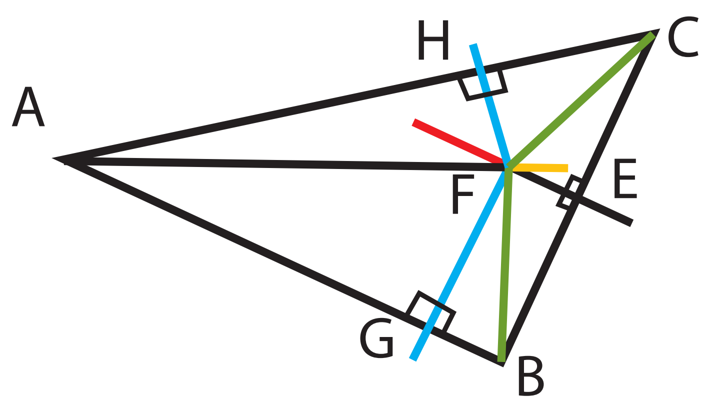
But what he forgot to say was that inccorect figures can lead to incorrect results, even with correct reasoning! A case in point is

Proposition: All triangles are isosceles.

Proof: Given a triangle $\Delta(A, B, C)$, if the line which bisects the angle at A meets the side BC at right angles (at the point D), then the triangles $\Delta(BAD)$ and $\Delta(CAD)$ are congruent, since $\angle(BAD) = \angle(CAD)$, $\overline{AD} = \overline{AD}$ and $\angle(ADB) = \angle(ADC) = \pi/2$, so the conditions of the angle-side-angle theorem hold. Therefore $\overline{AB} = \overline{AC}$ (corresponding parts of congruent polygons are congruent ('CPCPC')), and so $\Delta(ABC)$ is isosceles.



So, suppose that the angle bisector does not meet side BC in right angles. This means that the perpendicular bisector of the side BC and the angle bisector are not parallel, and so will meet at a point. Let E denote the midpoint of the side BC, and let the perpendicular bisector (through E) and the angle bisector meet at a point F inside of the triangle. (We no longer need (until we want to recover from our mistake!) to know where the angle bisector meets the side BC, so we will not mark where D is anymore.)



Now if we draw the lines through F perpendicular to the sides AB and AC, meeting these sides at point G and H, respectively, and draw in the line segments FB and FC, we have cut our original triangle into six smaller triangles. We now preceed to show that these triangles are congruent in pairs.

First, $\angle(GAF) = \angle(BAD) = \angle(CAD) = \angle(HAF)$, $\overline{AF} = \overline{AF}$, and $\angle(AFG) = \pi/2 - \angle(GAF) = \pi/2 - \angle(HAF) = \angle(AFH)$, so the triangles $\Delta(GAF)$ and $\Delta(HAF)$ are congruent by angle-side-angle, and so $\overline{GF} = \overline{HF}$ by CPCPC. Second, $\overline{BE} = \overline{CE}$ (since E is the midpoint of BC, $\overline{FE} = \overline{FE}$, and $\angle(BEF) = \pi/2 = \angle(CEF)$, so $\Delta(BEF)$ and $\Delta(CEF)$ are congruent by side-angle-side, so $\overline{BF} = \overline{CF}$ by CPCPC.

Finally, since $\angle(FHC) = \pi/2 = \angle(FGB)$ and $\overline{GF} = \overline{HF}$ and $\overline{BF} = \overline{CF}$ from the above, then $(\overline{GB})^2 = (\overline{BF})^2 - (\overline{GF})^2 = (\overline{CF})^2 - (\overline{HF})^2 = (\overline{HC})^2$, by Pythagoras, so $\overline{GB} = \overline{HF}$

(which, OK, is what we really need, but) so $\Delta(GFB)$ and $\Delta(HFC)$ are congruent by side-side-side. [Alternatively, there is a right_angle-side-side congruence theorem; the proof of it is essentially what we just did!]

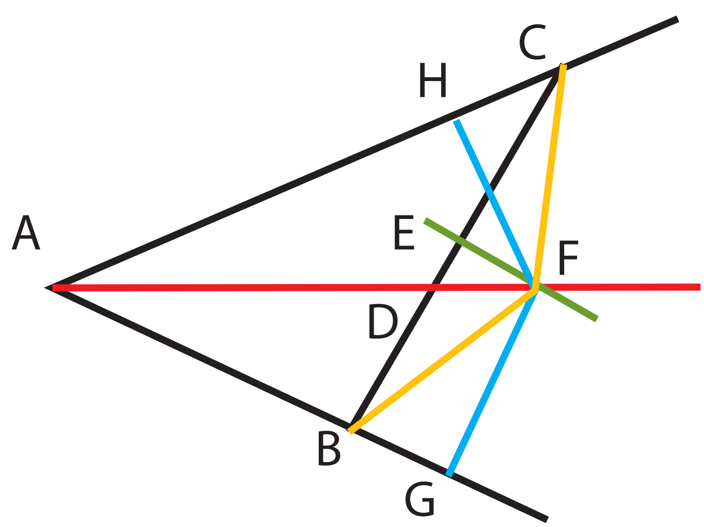
What this all means is that $\overline{AG} = \overline{AH}$, and $\overline{GB} = \overline{HC}$, and so

$$\overline{AB} = \overline{AG} + \overline{GB} = \overline{AH} + \overline{HC} = \overline{AC}.$$

So, $\Delta(ABC)$ is an isosceles triangle!

OK, so this is really isn't (always) true; in fact, if the angle bisector at A does not meet BC in a right angle, it is never true! What went wrong?

The answer is that it is the placement of the point F to lie inside of the triangle. This in turn traces to the true location of the point D. In this second part of our argument, an accurate drawing of the situation puts D on the 'other' side of E, so that the angle bisector and the perpendicular bisector always meet outside of the triangle. Every other argument we gave is in fact correct, except that some of the triangles we are looking at are outside of $\Delta(ABC)$. This has the effect, when we find that $(\overline{GB})^2 = (\overline{HC})^2$, that the correct conclusion to draw from this that $\overline{GB} = -\overline{HC}$ (in a technical sense), and that $\overline{AB} = \overline{AG} - \overline{GB}$, while $\overline{AC} = \overline{AH} + \overline{HC} = \overline{AG} + \overline{GB}$ (or the analogous statement with the minus-sign in the other equation). So the two side lengths are not equal; they differ by $2 \cdot \overline{GB}$ (!).



The point is that either $\angle(ADB)$ or $\angle(ADC)$ is less than $\pi/2$ (WOLOG, as in the figure, $\angle(ADB) < \pi/2$), and so $\angle(ABD) > \angle(ACD)$ (since $\angle(DAB) = \angle(DAC)$, and angles in a triangle add up to π), and so the Law of Sines gives

$$\frac{\sin(\angle(DAB))}{\overline{DB}} = \frac{\sin(\angle(ABD))}{\overline{AD}} > \frac{\sin(\angle(ACD))}{\overline{AD}} = \frac{\sin(\angle(DAC))}{\overline{DC}} = \frac{\sin(\angle(DAB))}{\overline{DC}}$$

which implies that $\overline{DC} > \overline{DB}$ and so E (the midpoint) lies to the 'C'-side of D; since $\angle(ADC) = \pi - \angle(ADB) > \pi/2$, this means that the angle bisector at A meets the perpendicular biscetor of BC outside of the the triangle $\Delta(ABC)$, every single time.

And $\sin(\angle(ABD)) > \sin(\angle(ACD))$, because, setting $\beta = \angle(ABD)$ and $\gamma = \angle(ACD)$, we have $0 < \gamma < \beta < \pi$, so $\sin\gamma > 0$ and $\cos\beta > \cos\gamma$ (since $\cos x$ is decreasing on $[0, \pi]$), and $0 < \cos\gamma$ (since $\gamma + \beta < \pi$, so $0 < \gamma < \pi/2$). Then $0 < \beta - \gamma < \beta < \pi$ implies that

$$0 < \sin(\beta - \gamma) = \sin\beta\cos\gamma - \sin\gamma\cos\beta < \sin\beta\cos\gamma - \sin\gamma\cos\gamma = [\sin\beta - \sin\gamma]\cos\gamma,$$

so $\sin(\beta) - \sin(\gamma) > 0$.