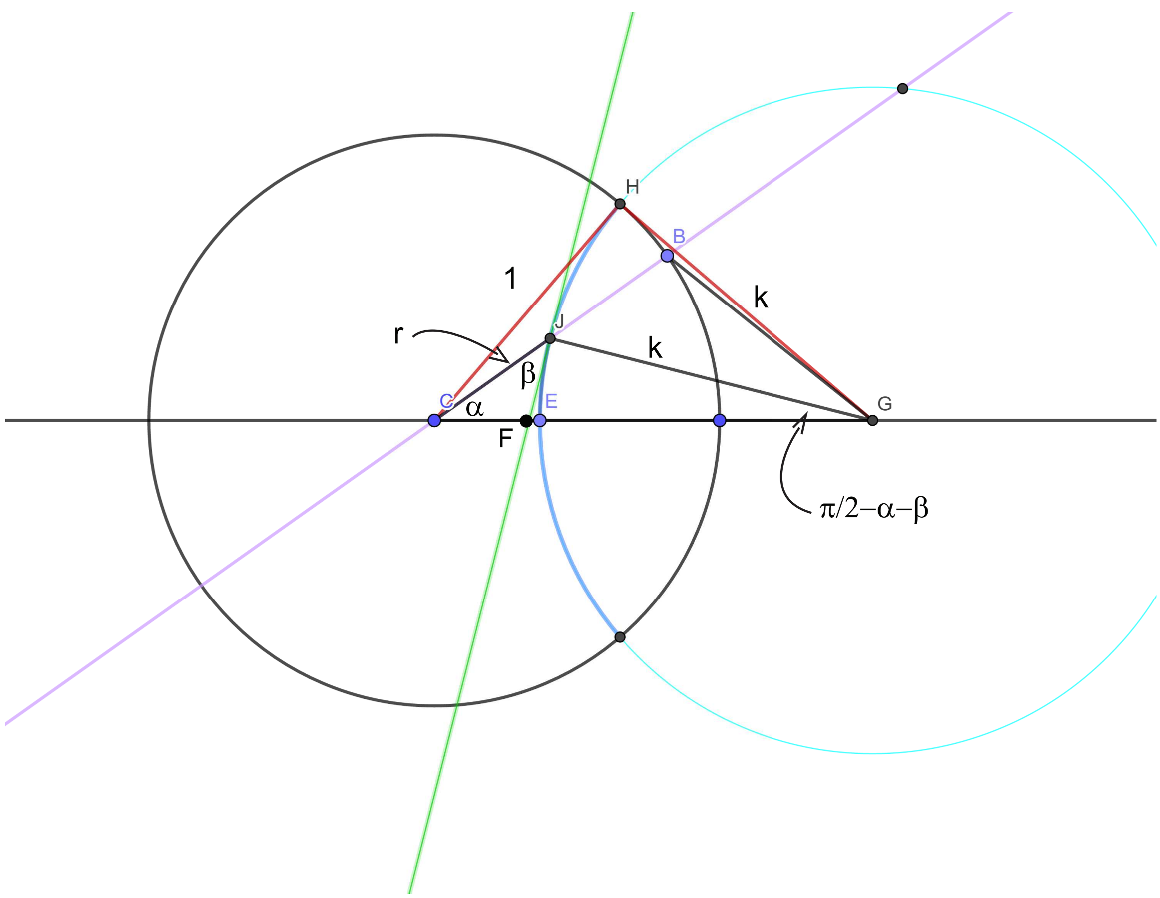
Hyperbolic area equals angle defect

In a hyperbolic right triangle with angles $\pi/2$, α and β, its angle defect Q is

$$Q = \pi - (\pi/2 + \alpha + \beta) = \pi/2 - \alpha - \beta.$$

If we fix the length of the side between α and the right angle (in the figure below, this is $|CE|$), and allow α to vary, then (by angle-side-angle congruence) β can be treated as a function of α, and so Q is a function of α. By a somewhat roundabout process, we can compute the derivative of Q with respect to α. This is the key step in establishing the result in the title.

We will work in the disc model of the hyperbolic plane. In the figure below (drawn initially using the GeoGebra program, with angle labels added) we have moved the vertex at α to the origin C for ease of exposition, and our hyperbolic right triangle has sides the line segments $\overline{CE}$ and $\overline{CJ}$ and third side the arc of the circle JE with center denoted G. The line at J is tangent to the circle, and so makes angle β with the line CJ. The circle centered at G through J and E is orthogonal, at H, to the circle S^1_∞ of radius 1. We let k denote the (*Euclidean*) radius of the circle at G containing the arc JE, so in the figure $|GH| = |GJ| = |GE| = k$. Note that this number is constant throughout the argument; it is (only) the points J (and F and B (which we never use)) which change as α changes. Since the circle S^1_∞ and the circle centered at G are orthogonal at H, we have that their radii $\overline{CH}$ and $\overline{GH}$ are orthogonal, so the triangle $\Delta(CHG)$ is a right triangle and the (Euclidean) Pythagorean Theorem yields $|CG| = \sqrt{k^2+1} = \ell$, which is also constant throughout our argument.



Since the tangent at J and the radial arc $\overline{GJ}$ are perpendicular, this implies that the angle $\angle(GJC) = \beta + \pi/2$, and so the angle $\angle(JGC) = \pi - (\alpha + \beta + \pi/2) = \pi/2 - \alpha - \beta = Q$ (since Euclidean triangles have angle sum π). So what we wish to understand, therefore, is the rate of change of $Q = \angle(JGC)$ as a function of $\alpha = \angle(JCG)$.

But this can be worked out using Euclidean trigonometry! Let the line segment CJ have (Euclidean) length r (so r is an (increasing) function of α; consequently, we can treat α as a function of r). By the Law of Cosines, applied to $\Delta(CJG)$, we have

$$k^2 = r^2 + \ell^2 - 2r\ell\cos\alpha \text{ and so } \cos\alpha = \frac{r^2 + \ell^2 - k^2}{2r\ell} = \frac{r^2+1}{2r\ell} = \frac{1}{2\ell}(r + r^{-1})$$

Similarly, $r^2 = k^2 + \ell^2 - 2k\ell\cos Q$, so $\cos Q = \dfrac{k^2 + \ell^2 - r^2}{2k\ell} = \dfrac{2k^2+1}{2k\ell} - \dfrac{r^2}{2k\ell}$.

Also, by the Law of Sines, applied to $\Delta(CJG)$, we have $\dfrac{\sin\alpha}{k} = \dfrac{\sin Q}{r}$, so $r\sin\alpha = k\sin Q$.

We have now, essentially, expressed both $\cos\alpha$ and $\cos Q$ as functions of r, and so we can take derivatives (with respect to r)!

$\dfrac{d(\cos(\alpha))}{dr} = -\sin\alpha\dfrac{d\alpha}{dr} = \dfrac{d}{dr}(\dfrac{1}{2\ell}(r + r^{-1})) = \dfrac{1}{2\ell}(1 - r^{-2}) = \dfrac{r^2-1}{2r^2\ell}$, so

$\dfrac{d\alpha}{dr} = \dfrac{1-r^2}{2r^2\ell\sin\alpha}$, and

$\dfrac{d(\cos(Q))}{dr} = -\sin(Q)\dfrac{dQ}{dr} = 0 - \dfrac{2r}{2k\ell}$, so

$\dfrac{dQ}{dr} = \dfrac{2r}{2k\ell\sin Q} = \dfrac{r}{k\ell\sin Q}$.

Consequently, by the Chain Rule (!) we have

$$\frac{dQ}{d\alpha} = \frac{dQ}{dr}\Big/\frac{d\alpha}{dr} = \frac{r}{k\ell\sin Q}\frac{2r^2\ell\sin\alpha}{1-r^2} = \frac{r\sin\alpha}{k\sin Q}\cdot\frac{\ell}{\ell}\cdot\frac{2r^2}{1-r^2} = \frac{2r^2}{1-r^2}$$

But, on the other hand, thinking of the Area of $\Delta(CJE)$ as a function $A(\alpha)$ of the angle α, then $A(\alpha+\epsilon) - A(\alpha)$ is approximately the (hyperbolic) area of a circular sector with (Euclidean) radius r and angle ϵ, which is $\frac{\epsilon}{2\pi}$ times the area of a circle with (Euclidean) radius r. But we have seen that the (hyperbolic) circumference of the circle of radius r is $C = \frac{4\pi r}{1-r^2}$, and Euclidean radius r means hyperbolic radius (since the radial line segment is a geodesic) $L = \int_0^r \frac{2\,dt}{1-t^2} = 2arctanh(r)$, so $r = \tanh(\frac{L}{2})$ and $C = 2\pi\sinh(L)$ (by plugging this expression for r into C and simplifying). Then the area of the circle of (Euclidean) radius r is the area of the circle of (hyperbolic) radius L, which is

$$\int_0^L 2\pi\sinh(t)\,dt = 2\pi(\cosh L - 1) = 2\pi(\cosh(2arctanh(r) - 1) = \frac{4\pi r^2}{1-r^2},$$

by applying a hyperbolic double angle formula and simplifying (exercise!). Consequently, $A(\alpha+\epsilon) - A(\alpha)$ is approximately $\dfrac{\epsilon}{2\pi}\dfrac{4\pi r^2}{1-r^2} = \epsilon\dfrac{2r^2}{1-r^2}$. 'Letting ϵ go to zero' we have

$$\frac{dA}{d\alpha} = \frac{2r^2}{1-r^2} = \frac{dQ}{d\alpha} \text{ for all } \alpha.$$

Since $A = Q = 0$ when $\alpha = 0$, one of those lovely consequences of the Mean Value Theorem implies that $A(\alpha) = Q(\alpha)$ for all α. Therefore, area equals angle defect for any right hyperbolic triangle. But since any triangle can be realized as the union or (set-theoretic) difference of two right triangles, and both area and angle defect add/subtract under union/difference, area equals angle defect for every hyperbolic triangle.