Math 990 Hyperbolic Geometry and Topology
Problem Set 1

General notation: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ *and* $f_A(z) = \dfrac{az+b}{cz+d}$ *is the corresponding* $\mathbb{R}^2_+$*-isometry.* $d_{\mathbb{R}^2_+}(z_1,z_2)$ *is the hyperbolic distance function in* $\mathbb{R}^2_+$*, and* $d_{\mathbb{D}^2}(w_1,w_2)$ *is the hyperbolic distance function in* $\mathbb{D}^2$*.*

1. Show that if $z_1 = 0$ is the origin in $\mathbb{D}^2$ and $z_2 \in \mathbb{D}^2$, then
$$d_{\mathbb{D}^2}(z_1,z_2) = \int_0^{|z_2|} \frac{1}{1-t^2}\, dt = \frac{1}{2}\ln\left|\frac{1+|z_2|}{1-|z_2|}\right| \ (= \operatorname{arctanh}(|z_2|)\ ).$$

2. Show that for $w_1 \in \mathbb{D}^2$, the map $\varphi : z \mapsto \dfrac{z - w_1}{-\overline{w_1}z + 1}$ is an isometry of $\mathbb{D}^2$ sending w_1 to the origin. [Hint: use our particular isometry $\psi : \mathbb{D}^2 \to \mathbb{R}^2_+$ and show that $\psi \circ \varphi \circ \psi^{-1} = f_A$ for some $A \in SL(2,\mathbb{R})$.] Use this, together with the previous problem, to give an explicit formula for $d_{\mathbb{D}^2}(z_1,z_2)$ for all $z_1, z_2 \in \mathbb{D}^2$.

3. Use Prbolem #2 and the map ψ to give a fairly ugly explicit formula for $d_{\mathbb{R}^2_+}(z_1,z_2)$ for all $z_1, z_2 \in \mathbb{R}^2_+$.

4. Show that if z_1, z_2 are distinct points in $\mathbb{R}^2_+$, as are w_1, w_2, and $d_{\mathbb{R}^2_+}(z_1,z_2) = d_{\mathbb{R}^2_+}(w_1,w_2)$, then there is an $A \in SL(2,\mathbb{R})$ with $f_A(z_i) = w_i$ for all i. [Hint: you could brute-force this, and solve equations, or show that transitivity on point-direction pairs is enough to guarantee it?]

5. Show that if $\alpha_1 < \alpha_2 < \alpha_3$ and $\alpha_1' < \alpha_2' < \alpha_3'$ are all in $\mathbb{R} \cup \{\infty\}$ (where we interpret $x < \infty$ for all $x \in \mathbb{R}$), then there is an $A \in SL(2,\mathbb{R})$ with $f_A(\alpha_i) = \alpha_i'$ for all i. [Hint: do this (first) for $\alpha_1 = 0$, $\alpha_2 = 1$ and $\alpha_3 = \infty$. Or: 'just' treat it as a system of linear equations!]

6. The isometry of $\mathbb{R}^2_+$ given by $\varphi(z) = -\overline{z}$ is a 'reflection': it fixes the positive imaginary axis (a geodesic) pointwise and swaps the two sides of the axis. Call anything conjugate to this a reflection ($\psi = f_A \circ \varphi \circ f_A^{-1}$ fixes pointwise (check!) the image of the imaginary axis under f_A). Show that every elliptic or loxodromic isometry of $\mathbb{R}^2_+$ is a composition of two reflections. Is this also true for a parabolic isometry?

7. Show that the 'AAA Congruence Theorem' for triangles holds when some of the vertices of the triangle are ideal vertices. [Problem #5 is basically the all-ideal-vertices case. When one of them isn't ideal, move it to the origin!]

8. Generalize the 'area equals angle defect' result to all polygons. [What is the correct notion of 'angle defect'?]. Use this to compute the hyperbolic area of the all-right-angle hexagon, and then compute the hyperbolic area of a closed orientable surface of genus $g \geq 2$.