Math 990 Hyperbolic Geometry and Topology
Problem Set 2

9. A sphere with 4 punctures has a decomposition into four ideal triangles. Use this to describe all of the complete hyperbolic structures on a 4punctured sphere, in terms of "translation coordinates" along the edges of the triangles. How many independent parameters do we have?

10. Show that in hyperbolic 3-space $\mathbb{H}^3$ for any two geodesics γ_1, γ_2 that do not share a common endpoint on the sphere at infinity there is a third geodesic γ_3 that meets both γ_1 and γ_2 and is perpendicular to both. [As usual, pick an isometry to make them look as nice as possible, first...]

11. Show that every geodesic in $\mathbb{H}^3$ is the intersection of two totally geodesic planes in $\mathbb{H}^3$, and, conversely, that the intersection of two totally goedesic planes (if non-empty!) is a geodesic in $\mathbb{H}^3$.

12. Given $z_1, z_2, z_3, w_1, w_2, w_3 \in \mathbb{C}$, under what conditions is there a fractional linear transformation $f(z) = \dfrac{az+b}{cz+d}$ with $f(z_i) = w_i$ for $i = 1, 2, 3$? Under what conditions is the map unique? [Try to directly write one down and see when you fail...]

13. (We found all of the isometries.) Use the fact that the geodesics of $\mathbb{H}^3$ (in the upper halfspace model) are the semicircles perpendicular to $\mathbb{R}^2 \times \{0\}$ and vertical lines to show that the isometries of $\mathbb{H}^3$ are precisely the fractional linear transformations of Problem #12 (composed with $(x, y, z) \mapsto (-x, y, z)$ to get the orientation-reversing maps). That is, show that "behavior at a single point" determines an isometry uniquely, and all such behaviors are captured by these maps.

14. Prove that if M is any 3-manifold obtained by gluing the faces of ideal tetrahedra, and the link of every vertex of the associated complex (by including the vertices) is a torus, then the number of 1-cells in the (ideal) triangulation of M equals the number of 3-cells (i.e., tetrahedra) in the triangulation. [Hint: Think about counting (fractions of) things in the links of the vertices, knowing what the Euler characteristics should be.]

15. Find a decomposition of the complement X of the knot 6_3 in S^3 into ideal tetrahedra, and write down the shape parameters and shape gluing equations for a hyperbolic structure on X. (See Purcell's notes in the "public" directory, section 5.1, for a refresher on decomposing X, if you need it.)

16. In Purcell's notes, page 45, she gives the following ideal tetrahedral decomposition of the complement X of the knot 6_1:

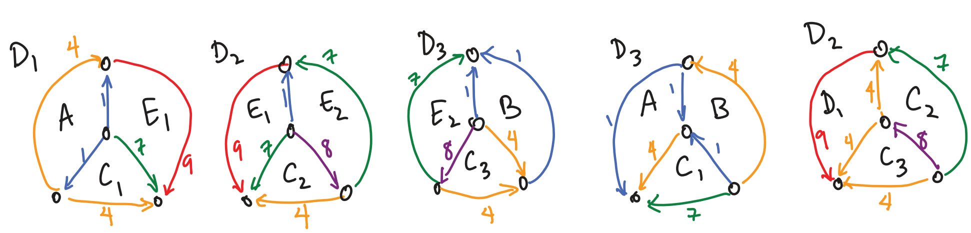

FIGURE 5.9. Five tetrahedra which glue to give the complement of the 6_1 knot.

Use this to give the shape gluing equations for the hyperbolic structure(s) on X. Draw the induced triangulation on the link of the ideal vertex, and give the corresponding completeness equations for a hyperbolic structure on X.