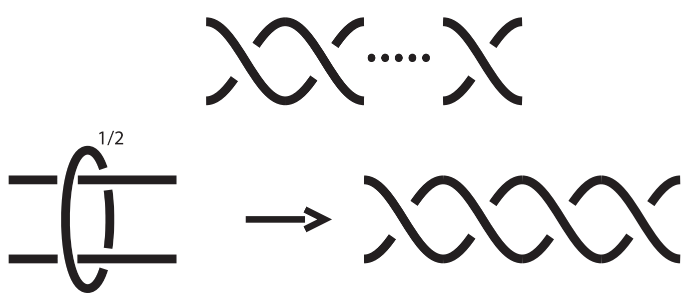
Math 990 Hyperbolic Geometry and Topology
Problem Set 3

17. (a) ["note this simplifying fact..."] If X is a (metric) space, $f, g, h : X \to X$ are isometries, and $f \circ g = g \circ f$, then setting $F = h \circ f \circ h^{-1}$ and $G = h \circ g \circ h^{-1}$ we have $F \circ G = G \circ F$.

    (b) Show that if $f, g \in Isom(\mathbb{H}^3)$ and $f \circ g = g \circ f$, then either (1) f and g are both hyperbolic and share the same axis, (2) f and g are both parabolic and share the same fixed point at infinity, or f and g are both elliptic, and either share the same axis or are both of order 2 and have axes that meet orthogonally. [Hint: (a)! Make one 'standard' and use their representations as matrices.]

18. Dylan Thurston (son of Bill!) showed that from a diagram of a knot K with k crossings you can tile the complement of K using k octahedra (with common north and south poles). [See a concise description of the construction in the first two pages of the paper "Bipyramids and bounds on volumes of hyperbolic links" by Colin Adams on the public webpage.] By cutting an octagon into tetrahedra, and arguing that a geodesic hyperbolic tetrahedron with some finite vertices must have volume no more than the regular ideal tetrahedron, give an upper bound on the volume of the complement of a hyperbolic knot with k crossings, in terms of k and v_3 = the volume of the regular hyperbolic ideal tetrahedron.

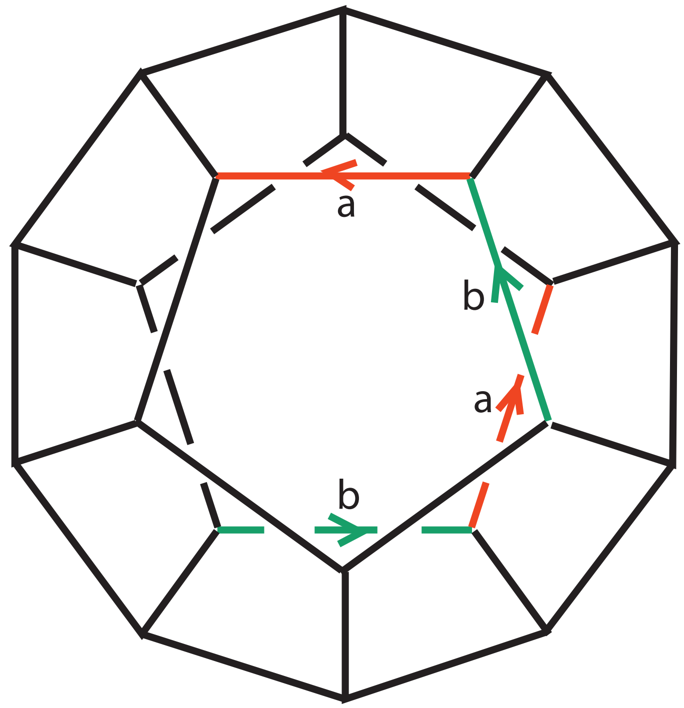
19. Given a knot diagram D for K, a *twist region* for the diagram is a collection of crossings (possibly one!) which form a single pair of twisted strands, as in the diagram. It is a fact that if a link (complement) $L = K \cup C$ has a component C bounded by a disk pierced twice by the remainder of the link K, then the result of Dehn filling along C with surgery coefficient $1/n$ results in the complement of the link K with n full twists added to the two piercing strands. The *twist number* $tw(K)$ of a knot is the minimum number of disjoint twist regions needed to encompass all of its crossings. Use Problem #18 and the fact that Dehn filling lowers volume to find an upper bound on the volume of a hyperbolic knot in terms of $t(K)$ and v_3.



20. Show that $1/n$ Dehn filling on the complement of the unknot (the knot with no crossings), i.e., filling so that $1 \cdot \mathcal{U} + n \cdot \lambda$ bounds a disk, always yields the 3-sphere S^3. [This is why in Problem #19 above, 'all' that happens when filling C is that the strands get twisted...]

20. The *Seifert-Weber dodecahedral space* is a 3-manifold obtained by gluing the opposite faces of a dodecahedron by a $3\pi/10$ twist. (See the figure for a depiction of one of these gluings.) Show that the resulting space is a (closed) 3-manifold, by verifying that the link of every vertex is a 2-sphere. If we treat the dodecahedron as a regular hyperbolic dodecahedron (i.e., the convex hull of symmmetrically placed points around the center), what must the dihedral angles be for the gluing consistency equations to provide a hyperbolic structure around the edges? If you want to dig more, show that there is a regular hyperbolic dodecahedron with those angles!

20. Play with the Seifert-Weber dodecahedral space! It is found in the SnapPy census as M = *DodecahedralOrientableClosedCensus(solids = 1)[-1]*. Try random retriangulations to make its Diriclet domain



look ‘dodecahedra-like’. Use SnapPy to (try to!) find a representation of the Seifert-Weber dodecahedral space as Dehn filling on a link in S^3; you can try to move towards a knot complement by Dehn drilling (and then looking at the ’also known as’ table), and filling and drilling... [Note: SnapPy reports that M has first homology $\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5$, which means that a link to Dehn fill to get it will need at least 3 components.] [N.B. I have not succeeded at this, myself, yet.]

21. Two manifolds M_1, M_2 are *commensurable* if there is a third manifold that is a finite-sheeted covering space of both. (The number of sheets need not be the same.) Show that “is commensurable to” is an equivalence relation. [N.B. This is ‘really’ a group theory question...]

22. (a) Show that if α, β are isometries of $\mathbb{H}^3$ that share no fixed point at infinity, then there is a third isometry γ so that $\gamma\alpha\gamma^{-1} = \alpha^1$ and $\gamma\beta\gamma^{-1} = \beta^{-1}$. [Hint: think about axes, and a rotation through pi around a well-chosen other axis! Or look in Thurston’s notes...]

    (b) (Show that!) the hypothesis of (a) is true for any pair of elements of the image $\varphi(\pi_1(M)) \subseteq PSL(2, \mathbb{C})$ of the fundamental group of a closed hyperbolic 3-manifold under the natural identification with a group of isometries, at least if α, β don’t both lie in an abelian subgroup. Conclude that for any word w=w(x,y) in the letters $x, y,$, if $w(\alpha, \beta) = 1$ in $\pi_1(M)$, then $w(\alpha^{-1}, \beta^{-1}) = 1$, as well. [N.B. Note that the same is actually true if they do lie in an abelian subgroup, but for a different reason!]

24. It is unknown if starting from a hyperbiolic alternating knot K and an alternating diagram D, changing some of its crossings can result in a hyperbolic knot K' with volume larger than the volume of K. It has, however, been verified that this cannot happen for a single crossing change, for alternating knots through 16 crossings (as of 2015). Devise a SnapPy experiment to randomly build alternating knots, randomly change some crossings, and test to see if the volume has gone up. Don’t forget to write any successes to a file, so that you can claim the prize of finding a counterexample!