4-MOVES AND THE DABKOWSKI-SAHI INVARIANT FOR KNOTS

MARK BRITTENHAM, SUSAN HERMILLER, AND ROBERT TODD

ABSTRACT. We study the 4-move invariant $\mathcal{R}_4(L)$ for links in the 3-sphere developed by Dabkowski and Sahi, which is defined as a quotient of the fundamental group of the link complement. We develop techniques for computing this invariant and show that for several classes of knots it is equal to the invariant for the unknot; therefore, in these cases the invariant cannot detect a counterexample to the 4-move conjecture.

1. INTRODUCTION

Studying the equivalence classes of knots and links under various types of transformations on their diagrams is a well-established subdiscipline of knot theory. This paper concerns the 4-move, first systematically studied by Nakanishi [Na2]. The 4-move belongs to the family of n-moves, which involve inserting or deleting n half-twists in series (see Figure 1), and is the only move in the family whose status as an unknotting operation has not yet been determined.

Theorem 1.1. [DP1],[DP2] *An n-move is an unknotting operation, i.e., every knot can be transformed to the unknot/unlink by isotopy and n-moves, if $n = 1, 2$. An n-move is not an unknotting operation if $n = 3$ or $n \geq 5$.*

In 1979 Nakanishi [Na1],[Ki, Problem 1.59 (3)(a)] conjectured that the 4-move is an unknotting operation. The conjecture remains open, though it has been verified for several classes of knots, including 2-bridge knots [Pr], 3-braids [Pr], and all knots with 12 or fewer crossings [DJKS]. Because of the theorem above, there is a growing belief that the conjecture is false. In fact, a leading candidate for a counterexample has emerged [As].

The search for a counterexample to the 4-move conjecture has focused on constructing knot and link invariants that are also invariant under 4-moves. Much of this work has used the fundamental group of the link complement, or of closely related spaces.
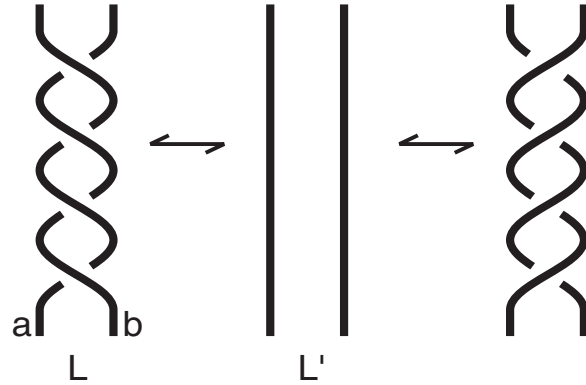
FIGURE 1. The 4-move

In [DS], Dabkowski and Sahi define an invariant of a link L in the 3-sphere, $\mathcal{R}_4(L)$, which is invariant under 4-moves. This invariant $\mathcal{R}_4(L)$ is a quotient of the fundamental group of the complement of L, $\pi(K) = \pi_1(\mathbb{S}^3 \setminus L)$, obtained by adding relations to a Wirtinger presentation of the link group. When L is the unknot then $\mathcal{R}_4(L) \cong \mathbb{Z}$; thus a counterexample to the 4-move conjecture can be found by finding a knot K with $\mathcal{R}_4(K) \not\cong \mathbb{Z}$. In what follows we say that a 4-move invariant is *trivially valued* for a knot K, if the invariant for the knot K takes the same "value" that the invariant takes on the unknot. That is, we will find a counterexample to the 4-move conjecture when we find an invariant and a knot for which the invariant is *not* trivially valued.

In this paper we define a new 4-move invariant for a knot K, $\mathcal{S}_4(K)$, as a quotient of $\mathcal{R}_4(K)$. We show that this new invariant is equally as strong a tool when looking for a counterexample to the 4-move conjecture, in the following.

Corollary 4.7. *The following are equivalent:*

(1) $\mathcal{R}_4(K) \cong \mathbb{Z}$; i.e., $\mathcal{R}_4(K)$ *is trivially valued*
(2) $\mathcal{S}_4(K)$ *is finite*
(3) $\mathcal{S}_4(K)$ *is abelian*
(4) $\mathcal{S}_4(K) \cong \mathbb{Z}_2$; i.e., $\mathcal{S}_4(K)$ *is trivially valued.*

Analysis of the invariant $\mathcal{S}_4(K)$ is more tractable than the group $\mathcal{R}_4(K)$. For any knot K the group $\mathcal{S}_4(K)$ is a quotient of a Coxeter group, namely a group generated by finitely many involutions, such that any pair of the generators generates the dihedral group of order 8. Analyzing subgroups generated by three of these involutions leads to the following.
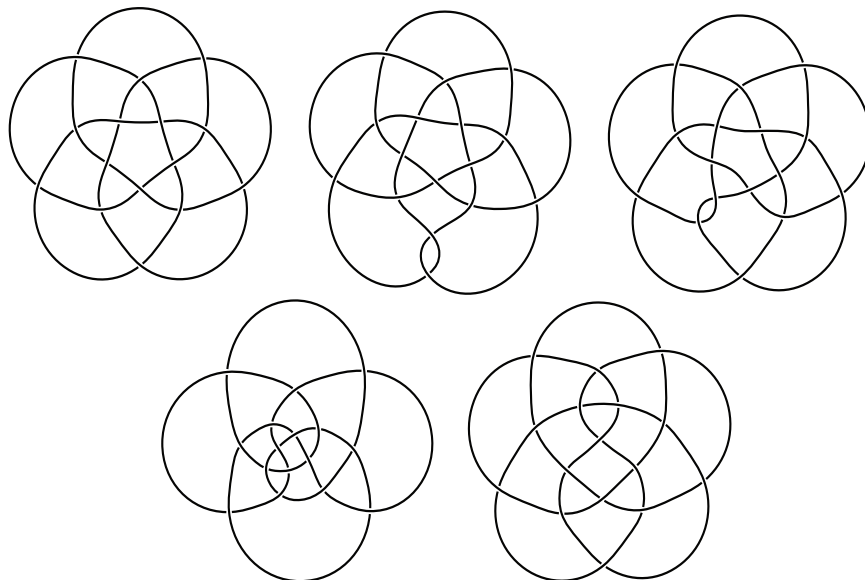
FIGURE 2. The smallest potential non-trivial examples

Theorem 4.8. *If K is a knot with bridge number 3, then $\mathcal{S}_4(K)$ (and thus $\mathcal{R}_4(K)$) is trivially valued. More generally, if $\pi(K)$ is generated by 3 meridians, then $\mathcal{S}_4(K)$ and $\mathcal{R}_4(K)$ are trivially valued.*

Another advantage to $\mathcal{S}_4(K)$ is that it is more amenable to algorithmic methods than $\mathcal{R}_4(K)$; computational software is often able to determine when the invariant $\mathcal{S}_4(K)$ is trivially valued, in cases that the corresponding computations applied to $\mathcal{R}_4(K)$ fail. In fact, via computer calculations, we determined that, for at least 99.9% of the 489,107,644 alternating knots with twenty crossings or less, $\mathcal{S}_4(K)$, and thus also the Dabkowski-Sahi invariant $\mathcal{R}_4(K)$, is trivially valued.

Theorem 5.1. *Among the alternating knots K with up to 20 crossings, $\mathcal{S}_4(K)$ (and thus $\mathcal{R}_4(K)$) is trivially valued, except possibly for 1 knot with 15 crossings, 4 knots with 16 crossings, 41 knots with 17 crossings, and 173 knots with 18 crossings, 31,612 knots with 19 crossings, and 274,217 knots with 20 crossings.*

The first 5 of these knots are shown in Figure 2; Gauss codes for those with 17 or fewer crossings can be found in an appendix at the end of the paper. The Gauss codes for the remaining 18, 19, and 20 crossing knots can be obtained by contacting the authors.

The computations for alternating knots utilized the enumeration by Flint, Rankin, and Schermann [FRS1],[FRS2] and their corresponding census as Gauss codes, which are amenable to machine computation.

(In addition, the diagrams for Figure 2 were generated by their online program *Knotilus* [KNOT].) Availability of similar censuses for other classes of knots would allow application of the same techniques to those classes.

In outline, in Section 2 we review the definition of $\mathcal{R}_4(L)$. In Section 3 we define the group $\mathcal{S}_4(K)$ as the top quotient in a normal series for $\mathcal{R}_4(K)$ and show that it is an invariant of knots under 4-moves. In Section 4 we reduce the problem of showing that $\mathcal{S}_4(K)$ (and thus $\mathcal{R}_4(K)$) is trivially valued to showing $\mathcal{S}_4(K)$ is *either* finite or abelian. In Section 5 we describe several large-scale computations carried out by the authors, and in Section 6 we discuss further avenues of research.

2. The invariant $\mathcal{R}_4(L)$

In [DS] Dabkowski and Sahi construct a a quotient of the fundamental group of the exterior of the link L, $\mathcal{R}_4(L)$, that is invariant under 4-moves. Recall that the Wirtinger presentation for a link group can be obtained from a diagram of the link: the generators a_i are represented by loops running around the i-th unbroken strand of the diagram, following the righthand rule; this requires an a priori choice of orientation to each component of the link. Each crossing provides a relation - either $a_i a_k = a_k a_j$ or $a_k a_i = a_j a_k$, depending upon the handedness of the crossing - where the overstrand labeled k separates the understrands i and j . As a_k conjugates a_j to a_i (or a_i to a_j), the generators assigned to each component of the link are all conjugate to one another. In the case of knots, as considered here, all of the Wirtinger generators are then conjugate to one another.

Starting from a Wirtinger presentation for a link group $\pi(L)$, one may view the invariant $\mathcal{R}_4(L)$ as follows. Say there is a 4-move taking L to L' (see Figure 1). Since $\mathcal{R}_4(L) = \mathcal{R}_4(L')$ is a common quotient of both knot groups, a certain word in the generators of $\pi(L)$ must be trivial in $\mathcal{R}_4(L)$ since the corresponding element in $\pi(L')$ <u>is</u> trivial. Depending upon the orientation of the strands of the underlying link, the needed relator has the form

$$babab^{-1}a^{-1}b^{-1}a^{-1}$$

where a and either b or b^{-1} are Wirtinger generators, corresponding to the two bottom strands of the initial configuration of the simplifying 4-move.

To construct their quotient Dabkowski and Sahi [DS, p. 1266] start with a Wirtinger presentation $\pi(L) = \pi_1(S^3 \setminus L) = \langle X|R \rangle$ and add the relators

$$babab^{-1}a^{-1}b^{-1}a^{-1}$$

for all a and b that are conjugates of an element of $X \cup X^{-1}$; that is, all a, b in the set

$$\mathcal{C} = \{gxg^{-1} : g \in (X \cup X^{-1})^*, x \in X^{\pm 1}\}$$

(where $(X \cup X^{-1})^*$ denotes the words in the alphabet $X \cup X^{-1}$). Note that whenever Y is another Wirtinger generating set for $\pi(L)$, then the subset $\{hyh^{-1} : h \in \pi(L), y \in Y^{\pm 1}\}$ of $\pi(L)$ equals $\mathcal{C}$; that is, the conjugacy classes of the generating set and their inverses must be the same. This follows from the fact that any two Wirtinger genera-tors associated to the same component of L, no matter the underlying projection, are represented by freely homotopic loops in $S^3 \setminus L$, and therefore are conjugate in $\pi(L)$ (using the same orientations of the components). That is, $\mathcal{R}_4(L)$ does not depend on the initial Wirtinger presentation chosen, and hence is invariant under Reidemeister moves and so is an invariant of the link L. Dabkowski and Sahi then show [DS, Proposition 2.3] that $\mathcal{R}_4(L)$ is, up to isomorphism, unchanged by a 4-move.

Throughout this paper, we consider this invariant in the case of a knot K. In this case, all pairs of Wirtinger generators are conjugate in $\pi(K)$. A presentation for $\mathcal{R}_4(K)$, with infinitely many relators, is then given, beginning with a Wirtinger presentation $\langle X|R \rangle$ for $\pi(K)$, as

$$\mathcal{R}_4(K) = \langle X|R \cup R'' \rangle = \langle X|R \cup R' \rangle,$$

where $R'' := \{(cd)^2(dc)^{-2} \ : \ c, d \in \mathcal{C}\}$ and

$$R' := \{(ba)^2(ab)^{-2} \ : \ a \in X, b = ga^\epsilon g^{-1}, \epsilon = \pm 1, g \in (X \cup X^{-1})^*\}.$$

These relators are relations $abab = baba$ in the quotient group $\mathcal{R}_4(K)$.

If K is the unknot then $\mathcal{R}_4(K) \cong \mathbb{Z}$ (choose the projection with no crossings and corresponding Wirtinger presentation). Consequently, any knot K that is 4-move equivalent to the unknot must have $\mathcal{R}_4(K) \cong \mathbb{Z}$.

Thus, as mentioned above, one may search for a counterexample to the 4-move conjecture by looking for a knot K with $\mathcal{R}_4(K) \not\cong \mathbb{Z}$.

3. THE INVARIANT $\mathcal{S}_4(K)$

In this section we will focus on knots K, and gain a better under-standing of $\mathcal{R}_4(K)$ by constructing a tower of subgroups of $\mathcal{R}_4(K)$ and studying the intermediate quotients. This leads to the definition of the invariant $\mathcal{S}_4(K)$.

For the given presentation of $\mathcal{R}_4(K)$, just as for the Wirtinger pre-sentation of $\pi(K)$, the sum of the exponents of each relation is 0. Thus,

just as for $\pi(K)$, the abelianization of $\mathcal{R}_4(K)$ is $\mathbb{Z}$, as all generators are conjugate. It follows that if $\mathcal{R}_4(K)$ is cyclic, then $\mathcal{R}_4(K) \cong \mathbb{Z}$. However, one can further show the following.

Lemma 3.1. *The group $G = \mathcal{R}_4(K)$ is cyclic iff $G/Z(G)$ is cyclic, where $Z(G)$ is the center of G. More generally, $G = \langle X|R \rangle$ is cyclic iff for some subset $S \subseteq Z(G)$ the group $G' = \langle X|R \cup S \rangle$ is cyclic.*

Proof. If G is cyclic, then every quotient of G is cyclic, so G' is cyclic. On the other hand, if G' is cyclic, then $G/Z(G)$ is the quotient of a cyclic group, so $G/Z(G) = \langle x \rangle$ is also cyclic. Choose any element $\tilde{x} \in G$ which maps to x under the standard projection $p : G \to G/Z(G)$. The group G is then generated by $\tilde{x}$ and $Z(G)$; given $y \in G$, then $p(y) = x^n$ for some n, so that $p(y\tilde{x}^{-n}) = 1$, and thus $y\tilde{x}^{-n} = z \in Z(G)$, and $y = z\tilde{x}^n$. But since $\tilde{x}$ commutes with every element of $Z(G)$ (by definition), G is abelian, so G equals its abelianization, which as we have already remarked, is $\mathbb{Z}$. So $G = \mathcal{R}_4(K)$ is cyclic. $\qquad\square$

In essence, one may add any element of the center of $\mathcal{R}_4(K)$ to its set of relators without altering the cyclicity of the group. This provides more avenues to determine if $\mathcal{R}_4(K)$ itself is $\mathbb{Z}$. This observation leads us to look for central elements of $\mathcal{R}_4(K)$.

Proposition 3.2. *If $\pi(K) = \langle X|R \rangle$ is a Wirtinger presentation for the knot group of the knot K, and $\mathcal{R}_4(K) = \langle X|R \cup R' \rangle$ is the corresponding presentation for $\mathcal{R}_4(K)$, then for every $a, b \in X$ we have $a^4 = b^4$ in $\mathcal{R}_4(K)$. In particular, for every $a \in X$, $a^4 \in Z(\mathcal{R}_4(K))$.*

Proof. Since $\mathcal{R}_4(K)$ is generated by the Wirtinger generators $b \in X$, to show that a^4 is central it suffices to show that a^4 commutes with every element $b \in X$. This will follow from the stronger fact that for every pair of Wirtinger generators $a, b \in X$ we have $a^4 = b^4$ in $\mathcal{R}_4(K)$, since b^4 clearly commutes with b, so a^4 also commutes with b.

Consider the Wirtinger generators a, b, and c, appearing at a crossing as in Figure 3, with relation $ac = cb$ (or $bc = ca$ for the crossing of the opposite sign). It suffices to show that $a^4 = b^4$ in $\mathcal{R}_4(K)$ for this pair a, b, since by induction, traversing the knot from undercrossing to undercrossing, one finds a string of identities $a_1^4 = a_2^4 = a_3^4 = \ldots$ for the successive Wirtinger generators, showing that all fourth powers of Wirtinger generators are equal.

Since our conclusion is symmetric in a and b it does not matter which Wirtinger relation actually holds; we will arbitrarily assume that $ac = cb$. For ease of reading we will follow common notational practice and set $A = a^{-1}$, $B = b^{-1}$, and $C = c^{-1}$.
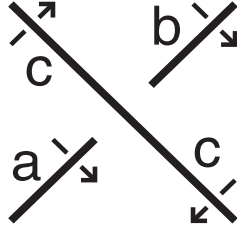
FIGURE 3. Wirtinger generators at a crossing

From $ac = cb$ it follows that $a = cbC$, and so $a^4 = (cbC)^4 = cb^4C$. Thus $a^4 = b^4$ so long as c commutes with b^4. Noting that c is also a Wirtinger generator, and so is conjugate to b, consider the following two relations from R':

$$(1) \qquad cbcb = bcbc$$
$$(2) \qquad c(bcB)c(bcB) = (bcB)c(bcB)c \ .$$

Relation (1) implies that b commutes with cbc, and thus B also commutes with cbc. Then

$$
\begin{aligned}
(bcbcbcb)cB^4 &= bcbcb[(cbc)(B^3)]B \overset{(1)}{=} bcbcb[(B^3)(cbc)]B = (bcbc)B^2cbcB \\
&\overset{(1)}{=} (cbcb)B^2cbcB = c(bcB)c(bcB) \overset{(2)}{=} (bcB)c(bcB)c \\
&= bc(bB^2)cbcBc = bcb[(B^2)(cbc)]Bc \overset{(1)}{=} bcbcbcB^3c \\
&= (bcbcbcb)(B^4c) \ .
\end{aligned}
$$

Canceling $(bcbcbcb)$ from both sides we get $cB^4 = B^4c$, so c commutes with B^4, and so c commutes with b^4, as desired. $\square$

This leads to the tower of subgroups of $\mathcal{R}_4(K)$. For the rest of this section, and section 4, fix a Wirtinger presentation $\pi(K) = \langle X|R \rangle$ of the knot group. The bottom of the tower for $\mathcal{R}_4(K)$ is the subgroup

$$N := \langle \{a^4 \ : \ a \in X\} \rangle$$

Since $a^4 \in Z(\mathcal{R}_4(K))$ for all $a \in X$, N is a normal subgroup of $\mathcal{R}_4(K)$, contained in the center of $\mathcal{R}_4(K)$. As $a^4 = b^4$ for all $a, b \in X$ by Proposition 3.2, $N = \langle a^4 \rangle$ is cyclic. Lemma 3.1 then says that $\mathcal{R}_4(K)$ is cyclic iff $\mathcal{R}_4(K)/N$ is cyclic; note that if $\mathcal{R}_4(K)/N$ is cyclic then $\mathcal{R}_4(K)/N \cong \mathbb{Z}_4$, since in this case $\mathcal{R}_4(K) = \langle a| \rangle \cong \mathbb{Z}$.

The top subgroup of the tower for $\mathcal{R}_4(K)$ is the normal subgroup

$$H := \langle \{a^2 : a \in X\} \rangle^N$$

(where $\langle \ \rangle^N$ denotes the normal subgroup generated by the set) of $\mathcal{R}_4(K)$ generated by the squares of the images of the Wirtinger generators of $\pi(K)$. Since $a^4 = (a^2)^2$, we have $N \leq H$, giving the normal series

$$\mathcal{R}_4(K) \ \triangleright \ H \ \triangleright N \ \triangleright \ \{1\}$$

Recall that the set $\mathcal{C} = \{gag^{-1} : g \in (X \cup X^{-1})^*, a \in X^{\pm 1}\}$ is independent of the choice of Wirtinger generating set X for $\pi(K)$. The groups in this normal series can also be written as $N = \langle\{c^4 \ : \ c \in \mathcal{C}\}\rangle$ and $H = \langle\{c^2 \ : \ c \in \mathcal{C}\}\rangle$; therefore the subgroups H and N of $\mathcal{R}_4(K)$ are also independent of the choice of knot projection used for the Wirtinger presentation.

Definition 3.3. *For a knot K with Wirtinger presentation $\pi(K) = \langle X|R\rangle$ we set $\mathcal{S}_4(K) = \mathcal{R}_4(K)/H = \langle X|R \cup R' \cup S\rangle$ where $S = \{a^2 : a \in X\}$.*

From the discussion above, the following is immediate.

Proposition 3.4. *The group $\mathcal{S}_4(K)$ is an invariant of the knot K, and is invariant (up to isomorphism) under 4-moves.*

Note that if $\mathcal{R}_4(K) \cong \mathbb{Z}$, then under this isomorphism we have $N = 4\mathbb{Z}$ and $H = 2\mathbb{Z}$. In particular, $\mathcal{S}_4(K) = \mathcal{R}_4(K)/H \cong \mathbb{Z}_2$ is a finite, abelian 2-group in this case. The group $\mathcal{S}_4(K)$ is trivially valued if $\mathcal{S}_4(K) \cong \mathbb{Z}_2$.

In general note that in $\mathcal{S}_4(K)$ every generator $a \in X$ has order 2, and so we have $a = a^{-1}$. Moreover, each conjugate of a must also have order 2, and hence equals its own inverse. Therefore in the presentation of $\mathcal{S}_4(K)$ in Definition 3.3, the relations $babab^{-1}a^{-1}b^{-1}a^{-1}$ from R' can be replaced by relations $(ba)^4$ for all $a \in X$ and b conjugate to a. In particular, $\mathcal{S}_4(K)$ is a quotient of the Coxeter group $\langle X \mid \{a^2 = 1, (ab)^4 = 1 \mid a, b \in X\}\rangle$.

This enables us to write a more useful presentation for $\mathcal{S}_4(K)$, as:

Lemma 3.5. $\mathcal{S}_4(K) = \mathcal{R}_4(K)/H = \langle X|R \cup S \cup T\rangle, where\ T = \{(awaw^{-1})^4 : a \in X, w \in X^*\}$.

The obstruction to $\mathcal{R}_4(K)$ being cyclic, we shall show in the next section, lies in the top subgroup H of the tower. In particular, it is detected by the quotient group $\mathcal{S}_4(K)$.

4. FINITE $\mathcal{S}_4(K)$ IS SUFFICIENT

In this section we establish our main result, that for a knot K, the quotient $\mathcal{S}_4(K)$ is trivially valued (that is, is isomorphic to $\mathbb{Z}_2$) if and

only if $\mathcal{R}_4(K)$ is trivially valued (i.e., is isomorphic to $\mathbb{Z}$). This will be carried out in several steps. The first step relies on the following theorem of Baer.

Theorem 4.1. *([Ba]; see [Go], Ch. 3, Theorem 8.2) If $G = \langle X|U \rangle$ is a finite group whose generators X are conjugate in G, and if for every pair c, d in the conjugacy class containing X the subgroup $\langle c, d \rangle \leq G$ generated by c and d is a p-group, then G is a p-group.*

Proposition 4.2. *If $\mathcal{S}_4(K)$ is finite, then $\mathcal{S}_4(K)$ is a 2-group.*

Proof. In $\mathcal{S}_4(K) = \langle X|R \cup S \cup T \rangle$, for any c, d in the conjugacy class of X we have the relations $c^2 = 1$, $d^2 = 1$, and $(cd)^4 = 1$ in $\mathcal{S}_4(K)$. Hence the subgroup $\langle c, d \rangle \leq \mathcal{S}_4(K)$ is a quotient of the Coxeter group $\langle c, d|c^2, d^2, (cd)^4 \rangle$, which is the dihedral group of order 8, and hence is a 2-group. So the conditions of Baer's Theorem (with $p = 2$) are met, and $\mathcal{S}_4(K)$ is a 2-group. $\square$

Lemma 4.3. *If G is a finite 2-group, whose generators are conjugate to one another, then G is a cyclic group.*

Proof. This appears to be a standard result (we first learned of it from a discussion on Math Overload [MO]); for completeness, the argument is included here.

   The Frattini subgroup $F(G)$ of G is the set of all of the 'non-generators' of G, that is, all c such that if $b_1, .., b_n, c$ generate G then $b_1, .., b_n$ generate G. (See, e.g., [Ha] for basic properties of $F(G)$.) $F(G)$ is a normal subgroup of G, and $G/F(G)$ is an elementary abelian 2-group. But an elementary abelian 2-group is a direct sum of copies of the group $\mathbb{Z}_2$. Since G is generated by a single conjugacy class, the abelianization of G is cyclic, and so the quotient $G \to G/F(G) \cong (\mathbb{Z}_2)^n$ factors through a cyclic group. Thus $n = 1$. Choosing an element $t \in G$ that maps to a generator of $G/F(G)$, G is then generated by t and the finite set $F(G)$. But then from the definition of $F(G)$, every element of $F(G)$ can be inductively removed from the generating set, implying that G is generated by $\{t\}$, i.e., G is cyclic. $\square$

Corollary 4.4. *If $\mathcal{S}_4(K)$ is finite, then $\mathcal{S}_4(K) \cong \mathbb{Z}_2$.*

Proof. From Proposition 4.2 and Lemma 4.3, we have that $\mathcal{S}_4(K)$ is a cyclic 2-group; thus we must show that $\mathcal{S}_4(K)$ is not the trivial group. The abelianization of $\mathcal{S}_4(K)$ has presentation $\langle X \mid R \cup S \cup T \cup \{aba^{-1}b^{-1} \mid a, b \in X\} \rangle$, using the notation from Lemma 3.5. Now the commutator relations imply that the Wirtinger relations R can be replaced by relations $a = b$ for all $a, b \in X$, and the relations T

are all redundant. Hence the abelianization of $\mathcal{S}_4(K)$ has presentation $\langle a \mid a^2 \rangle$, and is the group $\mathbb{Z}_2$. □

Next we turn our analysis to the middle quotient H/N of our normal series.

Lemma 4.5. *H and H/N are abelian.*

Proof. Recall that H is the normal closure in $\mathcal{R}_4(K)$ of the squares of the images in $\mathcal{R}_4(K)$ of the Wirtinger generators X. H is therefore generated by the (possibly infinite) collection of conjugates of squares of Wirtinger generators of $\mathcal{R}_4(K)$. Given a pair of these generators x, y of H, we can then set $x = ga^2g^{-1}$ and $y = hb^2h^{-1}$ for some $a, b \in X$ and $g, h \in \mathcal{R}_4(K)$. Now set $p = gag^{-1}$ and $q = hbh^{-1}$; then $x = p^2$ and $y = q^2$. Write P for p^{-1} and Q for q^{-1}. Note that p and q are conjugate to elements of X.

Using the relations in $\mathcal{R}_4(K)$

$$
\begin{aligned}
&(1) \quad QpQp = pQpQ \text{ (so } pq = qPqpQp) \\
&(2) \quad qPqP = PqPq \text{ (so } qPq = PqPqp) \\
&(3) \quad qpqp = pqpq,
\end{aligned}
$$

we then find that

$$
\begin{aligned}
xyx^{-1}y^{-1} &= p(pq)qPPQQ \overset{(1)}{=} p(qPq)pQ(pq)PPQQ \\
&\overset{(2),(1)}{=} p(PqPqp)pQ(qPqpQp)PPQQ = qPqpqpQPQQ \\
&\overset{(3)}{=} qPpqpqQPQQ = 1
\end{aligned}
$$

Therefore, in H, $xyx^{-1}y^{-1} = 1$, i.e., $yx = xy$. Since this holds for every pair of generators of H, H is abelian. □

We now have the tools to establish the relationship between $\mathcal{S}_4(K)$ and $\mathcal{R}_4(K)$.

Theorem 4.6. *If K is a knot, and if $\mathcal{S}_4(K)$ is finite, then $\mathcal{R}_4(K) \cong \mathbb{Z}$.*

Proof. Since $\mathcal{S}_4(K) = \mathcal{R}_4(K)/H$ is finite, H is a finite index subgroup of the finitely generated group $\mathcal{R}_4(K)$ and so is finitely generated. Thus the generating set $C = \{ga^2g^{-1} : a \in X, g \in \mathcal{R}_4(K)\}$ for H contains a finite generating set for H. Notice that in H/N, every element of C has order 2, since $(ga^2g^{-1})^2 = ga^4g^{-1} = 1$, given that $a^4 = 1$ in H/N. Thus H/N is a finitely-generated abelian group, whose generators z_i all have order 2. It follows that H/N is isomorphic to a quotient of

$(Z_2)^n$ for some n (the map sending the ith standard generator of $(Z_2)^n$ to z_i is a surjective homomorphism). In particular, H/N is finite with order a power of 2. But then since $\mathcal{S}_4(K) \cong \mathbb{Z}_2$ by Corollary 4.4, then $\mathcal{R}_4(K)/N$ is a finite group with order a power of 2, and so $\mathcal{R}_4(K)/N$ is a finite 2-group. Since it is generated by the Wirtinger generators, all of which are conjugate, we conclude from Lemma 4.3 that $\mathcal{R}_4(K)/N$ is cyclic. Since $N \subseteq Z(\mathcal{R}_4(K))$, Lemma 3.1 implies that $\mathcal{R}_4(K)$ is cyclic, and so $\mathcal{R}_4(K) \cong \mathbb{Z}$. $\qquad\square$

Note that in general if $\mathcal{S}_4(K)$ is abelian, then $\mathcal{S}_4(K)$ equals its own abelianization, which was shown in the proof of Corollary 4.4 to be the group $\mathbb{Z}_2$. Since if $\mathcal{R}_4(K) \cong \mathbb{Z}$ then the quotient $\mathcal{S}_4(K) \cong \mathbb{Z}_2$ is both abelian and finite, we have the following.

Corollary 4.7. *The following are equivalent:*

(1) $\mathcal{R}_4(K) \cong \mathbb{Z}$; *i.e.,* $\mathcal{R}_4(K)$ *is trivially valued.*

(2) $\mathcal{S}_4(K)$ *is finite.*

(3) $\mathcal{S}_4(K)$ *is abelian.*

(4) $\mathcal{S}_4(K) \cong \mathbb{Z}_2$; *i.e.,* $\mathcal{S}_4(K)$ *is trivially valued.*

Corollary 4.7 demonstrates that the non-triviality of $\mathcal{R}_4(K)$ "lives" at the top stage of our filtration $\mathcal{R}_4(K) \triangleright H \triangleright N \triangleright \{1\}$. The main point to this result is that it appears in practice to be much easier to analyze the presentation of the group $\mathcal{S}_4(K)$ than that of $\mathcal{R}_4(K)$. For example, the fact that in $\mathcal{S}_4(K)$ every generator is its own inverse is in practice a great advantage.

As an example, a direct computation in GAP shows that the group
$$G_{3,5} = \langle a_1, a_2, a_3 \mid$$
$$\{a_i^2, (a_i(wa_jw^{-1}))^4 \ : \ i,j \in \{1,2,3\}, w \in \{a_1,a_2,a_3\}^*, |w| \le 5\}\rangle,$$
(where $|w|$ denotes word length) has order 5192, and in particular is finite. But for any knot group generated by at most three meridians, for example, any 3-bridge knot group, the group $G_{3,5}$ surjects onto $\mathcal{S}_4(K)$. In particular, the map that sends the elements a_i to the three generating meridians is a surjective homomorphism. This implies that $\mathcal{S}_4(K)$ is finite whenever $\pi(K)$ is generated by at most 3 meridians.

Theorem 4.8. *If K is a knot with bridge number 3, then $\mathcal{S}_4(K)$ (and thus $\mathcal{R}_4(K)$) is trivially valued. More generally, if $\pi(K)$ is generated by 3 meridians, then $\mathcal{S}_4(K)$ and $\mathcal{R}_4(K)$ are trivially valued.*

Przytycki has shown that all 2-bridge knots are 4-move equivalent to the unknot [Pr], so this computation provides no new information for bridge number less than 3. It is known that a knot whose group is

generated by two meridians is 2-bridge [BZ]; the corresponding result is not known to be true for three meridians.

It is tempting to continue this line of reasoning further; we can, for any n and k, define

$$G_{n,k} = \langle a_1, \ldots, a_n \mid$$
$$\{a_i^2, (a_i(wa_jw^{-1}))^4 \; : \; i, j \in \{1, \ldots, n\}, w \in \{a_1, \ldots, a_n\}^*, |w| \le k\rangle$$

and define the corresponding group $G_n = G_{n,\infty}$ as a direct limit of successive quotients, and ask:

Question 4.9. *Is G_n finite for all n?*

If the answer to this question is 'Yes', then $\mathcal{R}_4(K)$ is trivially valued (i.e., isomorphic to $\mathbb{Z}$) for all knots K. Note that G_n is finite iff $G_{n,k}$ is finite for some k, since if G_n is finite then it has a finite presentation. The generators and relations used in that finite presentation can be obtained from our given presentation by finitely many Tietze transformations; choosing k to be the length of the longest relation used, then $G_{n,k} \cong G_n$ is finite.

This implies that for any n for which the answer to the above question is 'Yes' we can in principle verify this answer by a finite computation. In particular, by running a parallel coset enumeration computation (with a staggered start) on the groups $G_{n,k}$ with ever larger k, for any of these groups G_n that are finite, the enumeration is guaranteed to terminate (see, for example, [HEO, Chapter 5] for details of this procedure).

5. LARGE-SCALE COMPUTATION OF $\mathcal{S}_4(K)$

The original goal of this project was to find a counterexample to the 4-move conjecture, by discovering a knot K for which $\mathcal{S}_4(K)$ was not $\mathbb{Z}_2$. No such example was found. (The title of this paper would otherwise have been quite different!) However, determining $\mathcal{S}_4(K)$ is in practice much more amenable to machine computation than determining $\mathcal{R}_4(K)$, particularly since to show that $\mathcal{S}_4(K) \cong \mathbb{Z}_2$ it suffices by Corollary 4.7 to show that $\mathcal{S}_4(K)$ is either abelian or finite. Such computations are much more likely to terminate, and are much quicker than computation of $\mathcal{R}_4(K)$.

Strictly speaking, one cannot hand to a program the infinite set of relators needed to describe $\mathcal{S}_4(K)$ or $\mathcal{R}_4(K)$; however, one may truncate the infinite set of relators. In analogy with the groups $G_{n,k}$ above, consider the groups $G_k(K)$ defined from a Wirtinger presentation $\pi(K) = \langle X \mid R \rangle$ for K by

$$G_k(K) := \langle X \mid R \cup S \cup R_k \rangle,$$

where $S = \{a^2 \mid a \in X\}$ and

$$R_k := \{(awbw^{-1})^4 \mid a, b \in X, w \in X^*, |w| \leq k\}.$$

Since every relator in this presentation of $G_k(K)$ is a consequence of the relators in the presentation of $\mathcal{S}_4(K)$ in Definition 3.3, there is a canonical surjective homomorphism $G_k(K) \twoheadrightarrow \mathcal{S}_4(K)$. Therefore, in order to show that $\mathcal{S}_4(K)$ is finite it suffices to show that $G_k(K)$ is finite for some k. By an identical argument to that given for the groups $G_{n,k}$ above, this is also a necessary condition for the finiteness of $\mathcal{S}_4(K)$.

For example, for the knot K described in [As] as a likely candidate for a counterexample to the 4-move conjecture, the computation that $\mathcal{S}_4(K) \cong \mathbb{Z}_2$, and therefore $\mathcal{R}_4(K) \cong \mathbb{Z}$, is almost immediate; its Wirtinger presentation, together with the relators a^2 and a small subset of the relators $(awbw^{-1})^4$ are sufficient for the program GAP to conclude that the resulting group $G_k(K)$, for all sufficiently large k, has order 2. (In fact, entering the presentation takes far longer than the computation!) Bolstered by such initial success, we carried out analogous computations on the largest census of knots at our disposal.

The input data needed for computing $G_k(K)$ that is specific to the knot K is the Wirtinger presentation, taken from a diagram of the knot. For alternating knots this presentation can be easily constructed from a Gauss/Dowker code for the knot [Ga]. Recall that the Gauss code of a knot diagram with n crossings is a string of the integers $\{1, \ldots, n\}$, each occurring exactly twice. The string is constructed by numbering the crossings 1 through n, and then traveling along the knot, writing down the crossing numbers encountered in order. For an alternating knot this is sufficient to construct a Wirtinger presentation for the knot, since the additional over/undercrossing information is strictly not needed; we can arbitrarily assume that the first crossing met is the overcrossing, knowing that succeeding crossings will alternate. (The 'incorrect' choice will lead to the mirror image of the knot, which has the same knot group.) Since the label for the crossing can be imputed to be the labeling for the overstrand at the crossing, we can determine which understrands meet at a crossing from the Gauss code. This is illustrated by the example in Figure 4.

As can be seen, the trio of numbers centered on an undercrossing (which we may arbitrarily assume occur at the entries of even index in the Gauss code sequence) reflect the generators, in order, in a Wirtinger relation. What they do not reflect is the exponent of the overcrossing generator, that is, the conjugator, in the relation. But since in our quotients $\mathcal{S}_4(K)$ and $G_n(K)$ we have $a_i^2 = 1$ for each generator and hence $a_i = a_i^{-1}$, one can replace a_i^{-1} with a_i in these defining relations

Gauss code :

    3,4,6,5,4,2,1,6,5,3,2,1

Wirtinger relations:

$$a_3 a_4^{-1} = a_4^{-1} a_6 \qquad a_6 a_5^{-1} = a_5^{-1} a_4$$
$$a_4 a_2 = a_2 a_1 \qquad a_1 a_6^{-1} = a_6^{-1} a_5$$
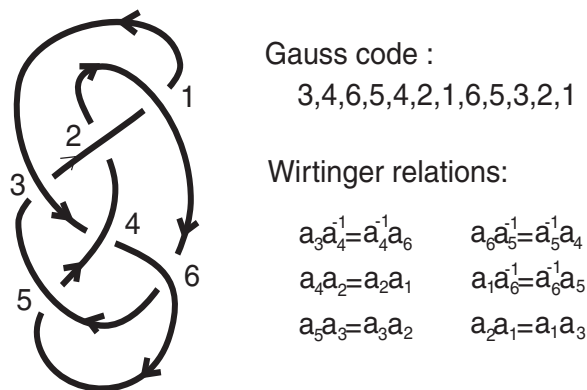$$a_5 a_3 = a_3 a_2 \qquad a_2 a_1 = a_1 a_3$$

FIGURE 4. From Gauss code to presentation

without changing the groups that are presented by these "Wirtinger relations" together with the relations $\{a_i^2 = 1 | a_i \in X\}$. These are the presentations that were extracted from Gauss codes for alternating knots, to use in our large-scale computations.

Flint and Rankin provide, on their website, the Prime Alternating Knot Generator software [PAKG] to generate the Gauss codes for every alternating knot (without duplication) of whatever number of crossings is specified by the user. From this, as described above, one can build a Wirtinger presentation for each knot, and then construct presentations for the groups $G_n(K)$ to test for finiteness.

Two basic algorithms were applied to analyze the group $G_n(K)$. First, a coset enumeration algorithm was used to enumerate cosets of the trivial subgroup. Second, we used the Knuth-Bendix algorithm (see [HEO, Chapter 12] for a description) to look for a confluent rewriting system, where the generators were given the order coming from the Gauss code, and words were given the "shortlex" order. In practice, four tests were applied in succession to winnow the initial list of alternating knots, eliminating those for which one of these tests determined that $G_n(K)$, and therefore $\mathcal{S}_4(K)$, was finite in turn. At each step, those knots for which $G_n(K)$ was not found to be finite were "failures" for that step, and the next step was applied just to those failure knots.

The general approach to the computations was the following sequence of steps:

(1) Apply a coset enumeration algorithm to enumerate the cosets of the trivial subgroup to $G_0(K)$. Collect failures.
(2) Apply the Knuth-Bendix algorithm to $G_0(K)$ for the failures from step 1. Collect failures.

(3) Apply the Knuth-Bendix algorithm to $G_1(K)$ for the failures in step 2. Collect failures.
(4) Apply the Knuth-Bendix algorithm to $G_2(K)$ for the failures in step 3. Collect failures.

Several software packages were used to perform these computations. The coset enumeration algorithm was applied via GAP [GAP] and its two implementations; the standard implementation via the "Size" command, and the GAP package ACE (Advanced Coset Enumeration). The Knuth-Bendix algorithm was applied via the package KBMAG (Knuth-Bendix on Monoids and Automatic Groups) [Ho] in GAP and via MAF [Wi], KBMAG's standalone PC implementation. In each case either a memory limit or a time limit was used to delineate success from failure. That is, either the computation finished or was abandoned after some limit was reached. In each case in which GAP (or one of its C++ packages, ACE or KBMAG) was used the standard memory limits (e.g. table size for coset enumeration) were used to delineate success from failure. In the cases where MAF was used, either a time limit was set (e.g. 5 minutes when considering $G_1(K)$), or an ad-hoc approach to determine that no progress was being made (this was done in step 4) was used to determine failure.

For alternating knots with 18 crossing or less all computations were performed on a personal computer, except for step 1 for 18 crossing knots, which were completed on *Firefly*, a 5600 core AMD cluster managed by the Holland Computing Center at the University of Nebraska at Omaha. In all of these cases, GAP's standard implementation of the coset enumeration algorithm was applied to the presentations of $G_0(K)$ via the "Size" command. Steps 2,3, and 4 were performed on a personal computer using MAF. After step 1, a relatively small list of these knots were left ($6681 = 8 + 82 + 1572 + 5019$ in total with 15 through 18 crossings). As noted above, in steps 2 and 3 a time limit of 5 minutes was set. After step 3 the list consisted of $763 = 1 + 27 + 201 + 534$ knots with 15 through 18 crossings. Thus, when applying step 4 to this small list, no time limit was set. In each of these cases, either the computation was successful or was halted by hand when the computation seemed to be making no progress (signaled by a long period of adding new equations without any reduction).

For the 19 and 20 crossing knots, steps 1 and 2 were performed on Tusker, a 40 TF cluster consisting of 106 Dell R815 nodes using AMD 6272 2.1GHz processors, also managed by the Holland Computing Center, at the University of Nebraska-Lincoln. Step 1 used the C++ coset

enumeration implementation ACE, that comes with the GAP installation (though GAP was not initiated as an interface) and step 2 used the C++ implementation of KBMAG that comes with the GAP installation (again, GAP was not used as an interface). After step 1 of the 80,689,811 19-crossing knots, all but approximately 450,000 were shown to have $G_0(K) \cong \mathbb{Z}_2$ and of the 397,782,507 20-crossing alternating knots, all but approximately 4,500,000 were shown to have $G_0(K) \cong \mathbb{Z}_2$. Step 2 reduced these numbers to 31,612 and 274,217, respectively. We have yet to apply steps 3 and 4 to the current list of failures for 19 and 20 crossing knots.

Theorem 5.1. *Among the alternating knots K with up to 20 crossings, $\mathcal{S}_4(K)$ (and thus $\mathcal{R}_4(K)$) is trivially valued, except possibly for 1 knot with 15 crossings, 4 knots with 16 crossings, 41 knots with 17 crossings, and 173 knots with 18 crossings, 31,612 knots with 19 crossings, and 274,217 knots with 20 crossings.*

Thus among the alternating knots with 20 or fewer crossings, all but at most 0.06% have trivially-valued Dabkowski-Sahi invariant. For those with 18 or fewer crossings, this percentage is 0.0021%. We anticipate that completion of steps 3 and 4 would significantly reduce the first percentage.

6. Concluding thoughts and future directions

Since every knot is 4-move equivalent to an alternating knot (choose crossings whose change would result in an alternating knot, and replace them with three crossings of the opposite sign), if there is a counterexample to the 4-move conjecture there is an alternating knot counterexample. One can view the above computations as either providing evidence in support of the 4-move conjecture or providing a larger set of possible counter examples, depending on one's own opinion of the truth or falsity of the 4-move conjecture. In fact, attempts by the authors to 4-move reduce some of the remaining knots identified by Theorem 5.1 to the unknot have never succeeded.

However, these computations may suggest that the invariants $\mathcal{R}_4(K)$ and $\mathcal{S}_4(K)$ cannot detect a counter-example to Nakanishi's 4-move conjecture. Thus we pose the following question:

Question 6.1. *Are $\mathcal{R}_4(K)$ and $\mathcal{S}_4(K)$ trivially valued for every knot K?*

The authors have formulated several lines of attack for this question; none as yet can be carried to their conclusion. For example, if one

could show that in $\mathcal{S}_4(K)$ every element has order dividing four then Question 6.1 could be answered in the affirmative, as $\mathcal{S}_4(K)$ would be the quotient of a finitely generated Burnside group of exponent 4, all of which are known [Sa] to be finite, and hence (by Corollary 4.7) $\mathcal{S}_4(K)$ would be isomorphic to $\mathbb{Z}_2$.

Ultimately, Question 6.1 is really a question of group theory. Given any group G that is normally generated by a single element x, then by Johnson's Theorem [Jo] this group is a quotient of a knot group via a surjection that sends a Wirtinger generator for the knot group (for some diagram) to x. However, if $G = \langle X \mid R \rangle$ is such a group, then one may consider the corresponding tower of subgroups constructed in Section 3 outside of the context of a knot diagram, namely

$$R(G) := \langle X|R \cup R'' \rangle \rhd H := \langle \{(gag^{-1})^2 : a \in X, g \in G\} \rangle^N$$
$$\rhd N := \langle \{(gag^{-1})^4 : a \in X, g \in G\} \rangle^N \rhd 1,$$

where $R'' = \{(cd)^2(dc)^{-2} \mid c, d \in \mathcal{C}\}$. That is, $R(G)$ is the quotient of G analogous to the quotient $\mathcal{R}_4(K)$ of $\pi(K)$, defined by modding out by the same set of relators. Since $\mathcal{R}_4(K)$, for any knot K built via Johnson's Theorem, would then surject onto $R(G)$, finding such a group G for which $R(G)$ is not cyclic would then imply the existence of a counterexample to the 4-move conjecture. As for knots, the cyclicity of $R(G)$ can be determined by the finiteness of $S(G) := R(G)/H$.

Thus one sees that Question 6.1 is equivalent to the following question:

Question 6.2. *Suppose that $G = \langle X|R \rangle$ with $|X| < \infty$ such that the set X lies in a single conjugacy class in G. If $S = \{x^2 : x \in X\}$ and $T = \{\{(xgyg^{-1})^4 : x, y \in X, g \in X^*\}$, is it always true that the quotient group $\langle X|R \cup S \cup T \rangle$ of G is cyclic? Equivalently, must this quotient be finite? Equivalently, must it be abelian?*

References

[As] N. Askitas, *A note on 4-equivalence*, J. Knot Theory Ramifications **8** (1999) 261-263.

[Ba] R. Baer, *Engelsche elemente Noetherscher Gruppen*, In: Invitations to geometry and topology, Math. Ann. **133** (1957) 256–270.

[BZ] M. Boileau and B. Zimmermann, *On the π-orbifold group of a link*, Math. Z. **200** (1989) 187–208.

[DS] M. Dabkowski and R. Sahi, *New Invariant of 4-moves*, J. Knot Theory Ramifications **16** (2007) 1261–1282.

[DJKS] M. Dabkowski, S. Jablan, N. Khan, and R. Sahi, *On 4-move equivalence classes of knots and links of two components*, J. Knot Theory Ramifications **20** (2011) 47–90.

[DP1] M. Dabkowski and J. Przytycki, *Burnside obstructions to the Montesinos-Nakanishi 3-move conjecture*, Geom. Topol. **6** (2002) 335-360.

[DP2] M. Dabkowski and J. Przytycki, *Burnside groups and rational moves*, preprint.

[FRS1] O. Flint, J. Schermann and S. Rankin, *Enumerating the Prime Alternating Knots, Part I*, J. Knot Theory Ramifications **13** (2004) 57-100.

[FRS2] O. Flint, J. Schermann and S. Rankin, *Enumerating the Prime Alternating Knots, Part II*, J. Knot Theory Ramifications **13** (2004) 101-149.

[GAP] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.12*; 2008, (`http://www.gap-system.org`).

[Ga] C.F. Gauss, *Werke, Band VIII* **272** Teubner, Leipzig, (1900) 282286.

[Go] D. Gorenstein, Finite groups, Chelsea Publ. Co., New York, NY, 1980.

[Ha] M. Hall, The theory of groups, Macmillan, New York, NY, 1959.

[Ho] D. Holt, KBMAG—*Knuth-Bendix in Monoids and Automatic Groups*, software package (1995), available from (`http://www.maths.warwick.ac.uk/~dfh/download/kbmag2/`).

[HEO] D. Holt, B. Eick, and E. O'Brien, *Handbook of Computational Group Theory*, Chapman and Hall, London, 2005.

[Jo] D. Johnson, *Homomorphs of knot groups*, Proc. Amer. Math. Soc. **78** (1980) 135-138.

[Ki] R. Kirby, *Problems in low-dimensional topology*, in Geometric Topology, Proceedings of the Georgia International Topology Conference, 1993, ed. W. Kazez, Studies in Advanced Mathematics, Vol. 2, Part 2 (AMS/IP, 1997) 35-473.

[KNOT] Knotilus, an online program and database for alternating knots and links, *http://knotilus.math.uwo.ca/*.

[MO] MathOverflow thread, October 2009, (`http://mathoverflow.net/questions/2650/`)

[Na1] Y. Nakanishi, *Foxs congruence modulo (2, 1)*, Sūrikaisekikenkyūsho Kōkyūroku **813** (1984) 102-110.

[Na2] Y. Nakanishi, *On Fox's Congruence Classes of Knots*, Osaka J. Math. **24** (1987) 217–225.

[PAKG] PAKG - Prime Alternating Knot Generator, available at *http://www.math.uwo.ca/s̃rankin/papers/knots/pakg.html*.

[Pr] J. Przytycki, *Topologia algebraica basada sobre nudos*, Proceedings of the First International Workshop on Graphs - Operads - Logic, (Cuautitlan, Mexico, 2001), arXiv:math.GT/0109029.

[Sa] I. Sanov, *Solution of the Burnside problem for exponent 4*, Uchen. Zap. Leningrad State Univ. Ser. Mat. **10** (1940) 166–170.

[Wi] A. Williams, *MAF (Monoid Automata Factory)*, software package (2009), available from (`http://maffsa.sourceforge.net/`).

APPENDIX A. PERL CODE FOR THE ALGORITHMS IN SECTION 5

I: Perl code computing the size of $G_0(K)$, calling GAP and ACE:

```perl
#!/usr/bin/perl
use strict; use warnings;
open ACE, "| /[absolute path to]/ace" or die print "can't open ace\n";
$"=",";  my @squares=(); my $xings=20;  # Don't forget to change this!!!
for my $i (1..$xings) { push(@squares,"$i^2"); }  my @prodpower4=();
for my $k1 (1..$xings) { for my $k2 (1..$xings)
{ push(@prodpower4,"(($k1)($k2))^4"); } }
for(my $i=197;$i<=200;$i++){  my $zero_num = sprintf("%04d", $i);
my $file="20xing".$zero_num;    # this is the filename for the codes
my $Stats="/work/unknots/rtodduno/20X/Stats.1/20xing.stats".$zero_num;
my $fail="/work/unknots/rtodduno/20X/Fail.1/20xing.fail".$zero_num;
my $cx="/work/unknots/rtodduno/20X/PCX.1/20xing.pcx".$zero_num;
open(ST,">>",$Stats);  print ACE "ao: $Stats;";
open(GC,"<$file") or die print "can't open $file\n";
my $s=1;  while(my $line=<GC>) { chomp $line;
my @knot=split(",",$line);  my $twoxings=2*$xings;  my @wirtinger=();
for my $j (1..$xings-1) {  my $overarc=$knot[2*$j-1];
my $dsarc=$knot[(2*$j)-2]; my $usarc=$knot[(2*$j)];
my $overgen="$overarc";    my $dsgen="$dsarc";
my $usgen="$usarc";    my $rel="($dsgen)($overgen)($usgen)($overgen)";
push(@wirtinger,$rel);  }
my $lastxing=$knot[$twoxings-1];  my $firstxing=$knot[0];
my $second2lastxing=$knot[$twoxings-2];
my $toprel="($second2lastxing)($lastxing)($firstxing)($lastxing)";
push(@wirtinger,$toprel);  my @allrelations=();
push(@allrelations,@squares);    push(@allrelations,@prodpower4);
push(@allrelations,@wirtinger);
my $command="Group Generators:$xings;Group Relators:
                                @allrelations;Subgroup: trivial;start;";
print ACE "$command \n";  $s++    }
close ST;  close GC;  open(ST,"<$Stats");  open(FAIL,">>",$fail);
open(PCX,">>",$cx);  my $t=1;
while(my $line=<ST>) { chomp($line); my @info=split(/ /,$line);
if($info[0] eq "OVERFLOW") { print FAIL "$t\n"; }
elsif($info[0] eq "INDEX") { if($info[2] != 2)
{ print PCX "$t\n"; } } $t++ }  close ST;  close FAIL;  close PCX;  }
close ACE;
```

II: Perl code computing an FCRS for $G_0(K)$, using KBPROG:

```perl
#!/usr/bin/perl
use strict;  use warnings;my $xings=20; # Don't forget to change this!!!
my @gens=();  for my $t (1..$xings)  { push(@gens,"f".$t.",F".$t) }
my @ingens=();  for my $q (1..$xings) { push(@ingens,"F".$q.",f".$q) }
$"=","; my @squares=();  for my $i (1..$xings)
{ push(@squares,"[f"."$i^2,IdWord]"); } my @prodpower4=();
for my $k1 (1..$xings) { for my $k2 (1..$xings)
{ push(@prodpower4,"[(f".$k1."*f".$k2.")^4,IdWord]"); } }
my $stillbad="stillbad".$ARGV[1];
open(SB,">>",$stillbad) or die print "can't open $stillbad\n";
open(GC,"<$ARGV[0]") or die print "can't open $ARGV[0]\n";
my $s=1; while(my $line=<GC>)
{ my $gapfile="20xingmaf".$ARGV[1]."/maf".$xings."_".$s.".txt";
  open(GAP,">>",$gapfile) or die print "can't open $gapfile \n";
  chomp $line; my @knot=split(",",$line); my $twoxings=2*$xings;
  my @wirtinger=(); for my $j (1..$xings-1)
{ my $overarc=$knot[2*$j-1]; my $dsarc=$knot[(2*$j)-2];
  my $usarc=$knot[(2*$j)]; my $overgen="f".$overarc; my $dsgen="f".$dsarc;
  my $usgen="f".$usarc;
  my $rel="[".$dsgen."*".$overgen."*".$usgen."*".$overgen.",IdWord]";
  push(@wirtinger,$rel); }
my $lastxing=$knot[$twoxings-1]; my $firstxing=$knot[0];
my $second2lastxing=$knot[$twoxings-2];
my $toprel="[f".$second2lastxing."*f".$lastxing."*f".$firstxing.
                                    "*f".$lastxing.",IdWord]";
push(@wirtinger,$toprel); my @allrelations=(); push(@allrelations,@squares);
push(@allrelations,@prodpower4); push(@allrelations,@wirtinger);
print GAP "_RWS:=rec(isRWS :=true,ordering";
print GAP " :=\"shortlex\",generatorOrder:=[@gens],";
print GAP "inverses:=[@ingens],equations:=[@allrelations]);";
system("/[absolute path to]/kbprog -silent $gapfile");   $s++;
my $outfile=$gapfile.".reduce"; my $extrafile1=$gapfile.".kbprog";
my $extrafile2=$gapfile.".kbprog.ec";
open(OUT,"<$outfile") or die print "can't open $outfile\n";  my @rec=<OUT>;
chomp $rec[12];   #  print "$rec[12]\n";
my $numst=substr($rec[12],-1);
if($numst !=2){ print SB "$line \n"; }
system("rm  $gapfile $outfile $extrafile1 $extrafile2");
}
```

Appendix B. Gauss codes for potential counterexamples to the 4-move conjecture

The alternating knots K with 17 or fewer crossings, for which the procedures of Section 5 have *not* shown that $\mathcal{R}_4(K) \cong \mathbb{Z}$:

1,2,3,4,5,6,7,8,9,3,10,11,6,12,13,9,2,14,11,5,15,13,8,1,14,10,4,15,12,7
1,2,3,4,5,6,7,8,9,10,11,12,6,1,13,9,14,15,12,5,2,13,8,16,15,11,4,3,10,14,16,7
1,2,3,4,5,6,7,8,9,10,11,5,12,1,8,13,14,11,4,3,15,9,13,16,6,12,2,15,10,14,16,7
1,2,3,4,5,6,7,8,9,10,11,12,6,13,2,9,14,15,12,5,16,3,10,14,8,1,13,16,4,11,15,7
1,2,3,4,5,6,7,8,9,10,11,5,12,1,8,13,14,11,4,15,2,9,13,16,6,12,15,3,10,14,16,7
1,2,3,4,5,6,7,8,9,10,11,12,13,14,6,15,4,11,16,17,8,1,15,5,12,16,10,3,2,9,17,13,14,7
1,2,3,4,5,6,7,8,9,10,11,12,13,14,8,3,2,9,15,16,12,5,6,13,17,15,10,1,4,7,14,17,16,11
1,2,3,4,5,6,7,8,9,10,11,12,13,14,8,1,15,5,12,11,4,3,16,9,14,17,6,15,2,16,10,13,17,7
1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,3,10,9,4,16,14,1,17,11,8,5,16,15,2,17,12,7,6,13
1,2,3,4,5,6,7,8,9,10,11,5,12,1,8,13,14,11,4,3,15,9,13,16,17,14,10,15,2,12,6,17,16,7
1,2,3,4,5,6,7,8,9,10,11,12,6,1,13,14,2,5,12,15,16,9,14,13,8,17,15,11,4,3,10,16,17,7
1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,9,16,1,6,13,12,5,2,16,8,17,14,11,4,3,10,15,17,7
1,2,3,4,5,6,7,8,9,10,11,12,6,5,13,14,2,15,8,16,12,13,17,3,15,9,10,1,14,17,4,7,16,11
1,2,3,4,5,6,7,8,9,10,11,12,6,13,14,3,2,15,8,16,12,5,17,14,15,9,10,1,4,17,13,7,16,11
1,2,3,4,5,6,7,8,9,10,11,9,12,13,6,14,15,3,10,11,2,16,14,5,17,12,8,1,16,15,4,17,13,7
1,2,3,4,5,6,7,8,9,10,8,11,12,5,13,14,2,15,11,16,6,13,17,3,15,9,10,1,14,17,4,12,16,7
1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,9,2,5,12,16,17,15,8,1,6,13,16,11,4,3,10,17,14,7
1,2,3,4,5,6,7,8,9,10,11,12,6,13,2,1,14,7,15,11,4,16,13,14,8,17,10,3,16,5,12,15,17,9
1,2,3,4,5,6,7,8,9,10,11,5,12,13,2,9,14,15,10,3,13,16,6,17,15,14,8,1,16,12,4,11,17,7
1,2,3,4,5,6,7,8,9,10,11,5,12,13,8,14,15,11,4,3,16,1,13,7,14,17,10,16,2,12,6,15,17,9
1,2,3,4,5,6,7,8,9,10,11,12,13,14,8,1,15,5,12,16,17,9,2,15,6,13,16,11,4,3,10,17,14,7
1,2,3,4,5,6,7,8,9,10,11,5,4,12,13,1,8,14,15,11,12,3,16,7,14,17,10,13,2,16,6,15,17,9
1,2,3,4,5,6,7,8,9,10,11,5,12,13,8,14,15,11,4,3,16,1,13,7,17,15,10,16,2,12,6,17,14,9
1,2,3,4,5,6,7,8,9,10,11,12,13,14,8,15,2,5,12,16,17,9,15,1,6,13,16,11,4,3,10,17,14,7
1,2,3,4,5,6,7,8,9,10,11,12,6,5,13,14,2,9,15,16,12,13,17,3,8,15,10,1,14,17,4,7,16,11
1,2,3,4,5,6,7,8,9,10,4,3,11,12,8,13,14,15,10,11,16,1,6,14,17,9,12,16,2,5,15,17,13,7
1,2,3,4,5,6,7,8,9,10,11,12,6,13,14,3,2,9,15,16,12,5,17,14,8,15,10,1,4,17,13,7,16,11
1,2,3,4,5,6,7,8,9,10,11,5,4,12,13,1,8,14,15,11,12,16,2,7,14,17,10,13,16,3,6,15,17,9
1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,9,2,3,10,16,12,5,17,1,8,15,16,11,4,17,6,13,14,7
1,2,3,4,5,6,7,8,9,10,11,12,13,5,14,15,16,1,10,13,4,17,15,7,8,16,2,11,12,3,17,14,6,9
1,2,3,4,5,6,7,8,9,10,11,3,12,5,13,9,14,1,15,12,4,11,16,14,8,17,6,15,2,16,10,13,17,7
1,2,3,4,5,6,7,8,9,10,11,12,2,13,6,14,8,15,12,3,16,5,10,17,15,1,13,16,4,11,17,9,14,7
1,2,3,4,5,6,7,8,9,10,11,5,12,13,2,14,10,15,6,12,4,16,14,1,17,7,15,11,16,3,13,17,8,9
1,2,3,4,5,6,7,8,9,10,11,5,12,3,13,14,10,7,15,12,4,16,14,1,17,15,6,11,16,13,2,17,8,9
1,2,3,4,5,6,7,8,9,10,11,12,4,13,2,14,10,7,15,5,12,16,14,1,17,15,6,11,16,3,13,17,8,9
1,2,3,4,5,6,7,8,9,10,11,12,8,13,6,14,15,3,10,11,2,16,14,5,17,9,12,1,16,15,4,17,13,7
1,2,3,4,5,6,7,8,9,10,11,12,4,13,14,1,10,7,15,5,12,16,2,14,17,15,6,11,16,3,13,17,8,9
1,2,3,4,5,6,7,8,9,10,11,3,12,7,13,14,10,15,16,1,8,13,6,17,4,11,15,16,2,12,17,5,14,9
1,2,3,4,5,6,7,8,9,10,11,3,12,5,13,14,8,1,15,12,4,11,16,9,14,17,6,15,2,16,10,13,17,7
1,2,3,4,5,6,7,8,9,10,11,3,12,5,13,9,14,1,15,12,4,16,10,14,8,17,6,15,2,11,16,13,17,7
1,2,3,4,5,6,7,8,9,10,11,3,12,7,13,14,15,11,2,16,8,13,6,17,4,15,10,1,16,12,17,5,14,9
1,2,3,4,5,6,7,8,9,10,11,12,4,13,6,14,8,15,2,11,16,5,13,17,15,1,10,16,12,3,17,7,14,9
1,2,3,4,5,6,7,8,9,10,11,12,4,13,14,7,8,15,2,11,16,5,17,14,15,1,10,16,12,3,13,17,6,9

1,2,3,4,5,6,7,8,9,10,11,3,12,5,13,14,8,1,15,11,4,12,16,7,14,17,10,15,2,16,6,13,17,9
1,2,3,4,5,6,7,8,9,10,11,5,12,13,2,14,10,7,15,12,4,16,14,1,17,15,6,11,16,3,13,17,8,9
1,2,3,4,5,6,7,8,9,10,11,3,12,5,13,7,14,1,10,15,4,12,16,14,8,17,15,11,2,16,6,13,17,9

Department of Mathematics, University of Nebraska, Lincoln NE 68588-0130, USA
  E-mail address: `mbrittenham2@math.unl.edu`

Department of Mathematics, University of Nebraska, Lincoln NE 68588-0130, USA
  E-mail address: `smh@math.unl.edu`

Department of Mathematics, University of Nebraska, Omaha NE 68182-2000, USA
  E-mail address: `rtodd@unomaha.edu`