ESSENTIAL LAMINATIONS AND BRANCHED SURFACES IN THE EXTERIORS OF LINKS

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§1 Introduction

Let M be a 3-manifold with a regular cell decomposition $\{B_k^3\}$. In [B], the first author shows:

> If M contains an essential lamination $\mathcal{L}_0$, then there is an essential lamination $\mathcal{L}$ in M, which is in normal form with respect to the cell decomposition $\{B_k^3\}$.

In the same paper, by using the result above, he proposes a procedure for determining whether a given manifold contains an essential lamination or not. However the procedure does not work, at present, since (1) there is not a practical algorithm for determining whether a given branched surface is essential or not, and (2) there does not exist an algorithm for determining whether a given branched surface fully carries a lamination or not.

The purpose of this paper is to try to carry out the procedure to the exteriors of links given by diagrams, by using various techniques in knot and link theory, and 3-dimensional topology. In fact, we give a definition of standard position (with respect to a diagram of a given link) for branched surfaces contained in the exterior of links in section 2, which is a natural generalization of standard position of closed incompressible surfaces defined by W. Menasco [M1]. In section 3, we apply the result of [B], to show that any essential lamination in a link exterior can be deformed into one carried by an essential branched surface in standard position with respect to a given diagram. In section 4, we study about branched surfaces in standard positions with respect to alternating diagrams, and give a sufficient condition for the branched surfaces to be incompressible and Reebless, and possess indecomposable exteriors (for the definitions of these terms, see section 2). In [O], U.Oertel studied some fundamental properties of affine laminations in 3-manifolds. In section 5, we give a necessary and sufficient condition for a given branched surface in standard position to fully carry affine laminations, by using

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admissible weights on train tracks obtained by the branched surface. We note that the results of sections 4 and 5 correspond to the above steps (1) and (2). They are not necessary and sufficient conditions. Conceivably, the conditions in section 4 are very far from necessary condition (see Example 6.2 of section 6), for branched surfaces to be incompressible and Reebless. However, we see that they are efficient enough to give a non-trivial example of a lamination with non-trivial holonomy in the figure eight knot complement.

§2 Preliminaries

For the definition of lamination, we refer to Chapter I of [MS]. For the definitions of essential lamination, and terms concerned with essential laminations, we refer to section 1 of [GO].

The notion of branched surfaces is defined in [FO]. Figure 2.1 shows a local model for a branched surface B and its corresponding fibred neighborhood $N(B)$ in a 3-manifold M. Each branch locus of a branched surface B is a circle or an arc properly immersed in M, and these branch loci are in general position in B, that is, intersecting each other transversely. Note that B has smooth structure near branch loci. There is a projection map $N(B) \to B$ which collapses every I-fiber of the I-bundle $N(B)$ to a point of B. The boundary $\partial N(B)$ is the union of three compact subsurfaces $\partial_h N(B)$, $\partial_v N(B)$ and $N(B) \cap \partial M$, which meet only in their common boundary points; every I-fiber of the I-bundle $N(B)$ meets $\partial_h N(B)$ transversely at its endpoints, while each I-fiber of $N(B)$ either is disjoint from $\partial_v N(B)$ or intersects $\partial_v N(B)$ in a union of at most two closed intervals in the interior of the fiber. Note that vertical boundary $\partial_v N(B)$ is also an I-bundle and is collapsed into the union of branch loci of B by the projection map.

$\boxed{\textit{Figure } 2.1}$

We recall the definition of essential branched surfaces in [GO]. A disk D properly embedded in $N(B)$ is called a *disk of contact* if D is transverse to the fibers and $\partial D \subset \operatorname{int} \partial_v N(B)$. A disk D properly embedded in $\mathrm{cl}(M - N(B))$ is called a *monogon* if $\alpha = \partial D \cap \partial_v N(B)$ is an I-fiber of $\partial_v N(B)$ and if $\partial D - \alpha \subset \partial_h N(B)$. A *Reeb branched surface* is a union of a torus T bounding a solid torus V and a meridian disk D of V which are glued at the branched locus $T \cap D = \partial D$ so that $\partial_v N(T \cup D) \subset \operatorname{int} V$. A branched surface B' is *carried* by B if $B' \subset N(B)$, and B' is transverse to the fibers of $N(B)$. A lamination $\mathcal{L}$ is *carried* by B if $\mathcal{L}$ is embedded in $N(B)$ and is transverse to the fibers. It is *fully carried* by B if $\mathcal{L}$ intersects every fiber of $N(B)$. A lamination $\mathcal{L}_R$

is a *Reeb lamination* if there is a solid torus V with a Reeb foliation $\mathcal{F}$ in M such that $\mathcal{L}_R$ is a union of leaves of $\mathcal{F}$ containing the toral leaf and at least one other.

A closed branched surface B is called *essential* if it satisfies the five conditions below.

(1) B has no disk of contact.
(2) No component of $\partial_h N(B)$ is a sphere, $\partial_h N(B)$ is incompressible in $\mathrm{cl}(M - N(B))$ and there is no monogon in $\mathrm{cl}(M - N(B))$
(3) $\mathrm{cl}(M - N(B))$ is irreducible and ∂M is incompressible in $\mathrm{cl}(M - N(B))$.
(4) B is Reebless i.e., B does not carry a Reeb branched surface.
(5) B fully carries a lamination.

Remarks.
(1) Suppose that M is an orientable, irreducible 3-manifold. Then a branched surface satisfying (1), (2) are called *incompressible* ([FO]).
(2) We say that the exterior of a fibered neighborhood of a branched surface is *indecomposable* if the branched surface satisfies (3)(see [GO, Remark 1.3]) .
(3) We will show in Appendix A that B satisfies the condition (4) above if and only if B does not carry a Reeb lamination.

It was shown by D. Gabai and U. Oertel that a lamination is essential if and only if there is an essential branched surface which fully carries the lamination ([GO, Proposition 4.5]). It is shown in [GO, Theorem 6.1] that if a compact orientable 3-manifold contains an essential lamination, then its universal cover is homeomorphic to $\mathbb{R}^3$. Y-Q. Wu showed that essential laminations in the exteriors of knots remain essential after majority of Dehn surgeries on the knots ([W]). Hence, to find an essential branched surface in a knot exterior is a very effective tactics on the study on Dehn surgeries on the knot. See, for example, [B], [B3], [DR], [Hay] and [HK].

Let L be a link in S^3, S the projection sphere, and E the diagram of L on S. We position L so that it lies on S except near crossings of E, where L lies on a "bubble" as shown in Figure 2.2. The inside of each bubble is called a *crossing ball.* Let S_+ (resp. S_-) be S with each disk of S inside a bubble replaced by the upper (resp. lower) hemisphere of that bubble. Let S_0 be S with interiors of the crossing balls are removed, i.e., $S_0 = S_+ \cap S_-$. Let B_+ (B_- resp.) be the ball in S^3 bounded by S_+ (S_- resp.) and lying above S_+ (below S_- resp.). A *region* of the diagram E is the closure of a component of $S_0 - E$.

Figure 2.2

Let F be a closed 2-manifold. A *train track* τ is a graph embedded in F such that each edge is smooth and has the same differential at each vertex. In this paper, we treat train tracks whose vertices have valency one or three. Let B' and B'' be branched surfaces in a 3-manifold M. We say that B'' is a *pinching* of B', or B' is a *splitting* of B'', if after an isotopy of B' and B'' there are neighborhoods $N(B')$, $N(B'')$ and an I-bundle J over a union of finitely many compact surfaces in M such that $N(B'') = N(B') \cup J$, where $J \cap N(B') \subset \partial J \cap \partial N(B')$, $\partial_h J \cap N(B') = \partial_h J \subset \partial_h N(B')$ and $\partial_v J \cap N(B')$ is empty or consists of finitely many components which are unions of fibers of $\partial_v N(B')$.

Let B be a closed branched surface in the exterior $E(L) = \mathrm{cl}(S^3 - N(L))$ of the link L. We say that B is in *standard position* with respect to the diagram E if B satisfies the next six conditions.

(1) B intersects each crossing ball in "saddle-shaped" disks as shown in Figure 2.2. In particular, crossing balls do not meet branch loci of B.

(2) B intersects S_+ (S_- resp.) transversely. Hence $B \cap S_+$ ($B \cap S_-$ resp.) are train tracks, say τ_+ (τ_- resp.).

(3) Each branch locus intersects S, i.e., no branch locus is entirely contained in $B_\pm$.

(4) There exists a union of finite number of mutually disjoint smooth disks $D_1^+ \cup \cdots \cup D_m^+$ ($D_1^- \cup \cdots \cup D_n^-$ resp.) properly embedded in B_+ (B_- resp.) and carried by $B \cap B_+$ ($B \cap B_-$ resp.) such that

(4-1) the branched surface $B \cap B_+$ ($B \cap B_-$ resp.) is a pinching of $D_1^+ \cup \cdots \cup D_m^+$ ($D_1^- \cup \cdots \cup D_n^-$ resp.), where no pinching occurs between subsurfaces of a single component of $D_1^+, \ldots, D_m^+$ ($D_1^-, \ldots, D_n^-$ resp.).

(4-2) The boundary of each D_i^+ (D_j^- resp.) meets a bubble.

(4-3) The boundary of each D_i^+ (D_j^- resp.) does not meet the same side of the bubble more than once.

(5) No arc component of (branch loci) $\cap B_\pm$ has its both endpoints in a region.

We call the system of disks in B_+ (B_- resp.) of the condition (4) of the above definition a *system of generating disks* for $B \cap B_+$ ($B \cap B_-$ resp.).

Remark on Condition (4-1). We note that no pinching occurs between subsurfaces of a single component of $D_1^+ \cup \cdots \cup D_m^+$ ($D_1^- \cup \cdots \cup D_n^-$ resp.) if and only if each D_i^+ (D_j^- resp.) is mapped to an embedded disk by the projection map $N(B \cap B_+) \to B \cap B_+$ ($N(B \cap B_-) \to B \cap B_-$ resp.), i.e., each fiber of $N(B \cap B_+)$ ($N(B \cap B_-)$ resp.) meets each D_i^+ (D_j^- resp.) in at most one point.

In fact, 'if' part of this assertion is clear. We can prove 'only if' part as follows. Since the argument is the same, we prove this only for $D_1^+ \cup \cdots \cup D_m^+$. Suppose that there exists a fiber, say J, of $N(B \cap B_+)$ intersecting D_i^+ more than once. Let J' be a subinterval of J such that $J' \cap D_i^+ = \partial J'$. Suppose $D_k^+ \cap J' \neq \emptyset$ $(k \neq i)$. Then we note that J' intersects D_i^+ in more than one points, since D_i^+, D_k^+ are mutually disjoint disks properly embedded in the 3-ball B_+. Hence by retaking D_i^+ if necessary, we may suppose that $J \cap (D_1^+ \cup \cdots \cup D_m^+) = J \cap D_i^+ = \partial J$. This shows that a pinching occurs between subsurfaces in D_i^+.

Remark on Condition (4).

We note that there is a branched surface B properly embedded in a ball such that B is a union of smooth disks and ∂B contains a smooth circle bounding no smooth disk in B. Here is an example:

We will construct B as a union of three smooth disks D_1, D_2 and D_3 properly embedded in a ball. We place these three disks to be parallel in the ball so that D_2 is between D_1 and D_3. We glue D_1 and D_2 in a half disk, then the union $D_1 \cup D_2$ is a branched surface with a single branch arc α. We glue D_2 and D_3 in a half disk, then the union $D_2 \cup D_3$ is a branched surface with a single branch arc β. Here we perform the pinching operations so that α and β intersect in two points on D_2 and that $D_1 \cap D_2 \cap D_3$ consists of two disks. Then the boundary of the branched surface $D_1 \cup D_2 \cup D_3$ is a union of two smooth circles glued in two subarcs. Then this boundary contains four smooth circles. But one of them does not bound a smooth disk in the branched surface.

$\boxed{\textit{Figure } 2.3}$

Furthermore, we can construct an example where no pair of subarcs of branch loci intersect each other more than one point. In the above branched surface, we take an arc γ in $(D_2 \cap D_3) - D_1$ connecting the arcs β and $(\partial D_2) \cap (\partial D_3)$. Let $N(\gamma)$ be a regular neighborhood of γ in the half disk $D_2 \cap D_3$. We split $D_2 \cup D_3$ along the band $N(\gamma)$. Then the branch arc β is deformed into two arcs each of which intersects α in a single point. But the boundary of $D_1 \cup D_2 \cup D_3$ still contains a smooth circle which does not bound a smooth disk.

This definition of standard position is a mimicry of that for closed surfaces in [M1]. However, we do not know whether every closed essential branched surface can be isotoped to be in standard position or not, while it can be after adequate splitting operations when E is connected (cf. Theorem 3.1).

A disk Δ embedded in the interior of a surface is a *0-gon* if $\partial\Delta$ is a smooth circle, and it is a *monogon* if $\partial\Delta$ is smooth except one corner point.

Lemma 2.1. *Let E, B, $\tau_\pm$ be as above. Suppose that B is in a standard position and that E is connected. Let Δ be a disk entirely contained in a region of E so that $\partial\Delta \subset \tau_\pm$. Then Δ is neither a 0-gon nor a monogon.*

Proof. Since the argument is the same, we prove this only for τ_+. Assume that there exists such a 0-gon or monogon, say Δ. An elementary calculation on Euler characteristics shows that the closure of a component of $\Delta - B$, say Δ' is a 0-gon, monogon or bigon. Let D' be the closure of the component of $S_+ - N(B)$ contained in Δ'. Let $Q_1, \ldots, Q_{2m}$ be duplicated parallel copies of $D_1^+, \ldots, D_m^+$ in $N(B \cap B_+)$ such that $(Q_1 \cup \cdots \cup Q_{2m}) \supset \partial_h N(B \cap B_+)$, where $D_1^+, \ldots, D_m^+$ are disks in the definition of standard position. Suppose $Q_k \cap \partial D' \neq \emptyset$. If Δ' is a 0-gon, then we have $\partial D_i^+ = \partial\Delta'$ for the disk D_i^+ parallel to Q_k, contradicting the condition (4-2) of the definition of standard position. If Δ' is a monogon, then we see that a pinching occurs between subsurfaces of the disk D_i^+ parallel to Q_k, contradicting the condition (4-1) of the definition of standard position. □

§3 Deforming an essential lamination to be in standard position

The goal of this section is the next theorem.

Theorem 3.1. *Let L be a link in the 3-sphere, E a diagram of L and $E(L)$ the exterior of L. Suppose that E is connected and that $E(L)$ contains an essential lamination $\mathcal{L}_0$ without boundary. Then $E(L)$ contains an essential lamination $\mathcal{L}$ without boundary such that it is fully carried by a closed essential branched surface B which is in standard position with respect to E.*

Moreover, we can take the lamination $\mathcal{L}$ so that $\mathcal{L}$ also remains essential in M, where M is any 3-manifold obtained by a Dehn surgery along the link L and containing $\mathcal{L}_0$ as an essential lamination. In particular, if the lamination $\mathcal{L}_0$ does not contain a toral leaf parallel to a component of the boundary $\partial E(L)$, then we can take $\mathcal{L}$ not to do.

First of all, we note that this theorem is based mainly on the result by the first author in [B]. A cell decomposition $\mathcal{C}$ of a 3-manifold is called *regular* if every k-cell is a polyhedron, and every face of every k-cell is glued to a $(k-1)$-cell by a homeomorphism. Let Z be a 3-cell of a regular cell decomposition $\mathcal{C}$ of a 3-manifold, and $\mathcal{C}_{\partial Z}$ the regular cell decomposition of the 2-sphere ∂Z induced from $\mathcal{C}$. Note that $\mathcal{C}_{\partial Z}$ may have two or more cells which are copies of the same cell of $\mathcal{C}$. A disk D properly embedded in Z is said to be *essential* if ∂D does intersect 1-skeleton of $\mathcal{C}_{\partial Z}$ transversely at one or more point. An essential disk D in Z is called a *normal disk* if ∂D intersects every 1-cell of $\mathcal{C}_{\partial Z}$ at no more than one point. Note that a normal disk may intersect a 1-cell of $\mathcal{C}$ in two or more points. Two normal disks in a 3-cell are in the same *type* if their boundaries cobound an annulus which is divided into rectangles by the 1-skeleton. A lamination $\mathcal{L} \subset M$ is in *normal form* with respect to the regular cell decomposition $\mathcal{C}$ if it is transverse to the decomposition, and it intersects the 3-cells in normal disks. A regular cell decomposition $\mathcal{C}$ of M is said to be *nice* in this paper if it has no 3-cell Z such that the induced cell decomposition $\mathcal{C}_{\partial Z}$ contains a pair of 2-cells sharing one or more 1-cells and amalgamated into a 2-cell of $\mathcal{C}$.

Then the following is known.

Theorem in [B]**.** *If M is a compact irreducible 3-manifold with a regular cell decomposition $\mathcal{C}$, and M contains an essential lamination $\mathcal{L}_0$ without boundary, then there is an essential lamination $\mathcal{L}$ without boundary in M which is in normal form with respect to $\mathcal{C}$.*

N.B. We give a warning that the regular cell decomposition $\mathcal{C}$ need to be nice in the above theorem because the ∂-compressing operation described in [B], page 4 below Figure 2 cannot be pursued if $\mathcal{C}$ is not nice.

When every 3-cell of the cell decomposition $\mathcal{C}$ is embedded in M, the same arguments as in [B4] will show the above theorem. Hence it is sufficient to read [B4] and the arguments below for understanding of Theorem 3.1.

We note that Theorem in [B] can be strengthened as in the following form.

Addendum to Theorem in [B]**.** *Let W be a compact 3-manifold containing M as a submanifold so that no component of $cl(W - M)$ contains more than one component of ∂M. Suppose that the original lamination $\mathcal{L}_0$ is essential in W. Then we can take the resultant lamination $\mathcal{L}$ to be essential also in W. We can take $\mathcal{L}$ to be the same one for all such 3-manifolds W.*

Proof. We assume good familiarity of the reader with [B]. We take a regular cell decomposition $\mathcal{C}'$ of W such that the restriction of $\mathcal{C}'$ on M is exactly the regular cell decomposition $\mathcal{C}$ of M. Then we apply the argument of [B] to $\mathcal{L}_0$ and $\mathcal{C}'$ to obtain an essential lamination $\mathcal{L}$ which is in normal form with respect to $\mathcal{C}'$. It is easy to see that the deformations for obtaining $\mathcal{L}$ from $\mathcal{L}_0$ stay in M. In fact, $\mathcal{L}$ can intersect a 3-cell C of $\mathcal{C}'$ only if $\mathcal{L}_0$ intersects C. We slightly change the operations described between Lemma and Proposition in section 4 in [B4] which is cited at right after Figure 9 in section 4 in [B]. There we discard certain sublaminations containing compressible toral leaves, but here we discard sublaminations containing separating toral leaves. (Note that we may assume the original lamination consists of non-compact leaves, and hence the toral leaves are split-and-paste leaves.) Since every component of $\mathrm{cl}(W - M)$ contains no more than one component of ∂M, a torus in M is separating if and only if it is separating in W. Hence we can obtain the essential lamination $\mathcal{L}$ also by applying the same arguments for $\mathcal{L}_0$ and $\mathcal{C}$ as for $\mathcal{L}_0$ and $\mathcal{C}'$. Thus we can see that $\mathcal{L}$ is essential also in M. □

We say that a branched surface B is in a *normal form* with respect to the regular cell decomposition $\mathcal{C}$ if

(1) The branch loci are disjoint from the 1-skeleton,
(2) B is transverse to $\mathcal{C}$,
(3) no branch locus is entirely contained in a 3-cell,
(4) B intersects every 3-cell in a pinching of the union of zero or more mutually disjoint smooth normal disks, where no pinching occurs between subsurfaces of a single disk,
(5) no arc of (a branch locus) $\cap$ (a 3-cell) has its both endpoints in a 2-cell.

Lemma 3.2. *Let $\mathcal{L}$ be an essential lamination without boundary in a normal form with respect to a regular cell decomposition $\mathcal{C}$ of a 3-manifold M. Then $\mathcal{L}$ is fully carried by a closed essential normal branched surface.*

Proof. First, we split leaves of $\mathcal{L}$ containing the highest or lowest normal disk of every type in every 3-cell (see lines 4–5 in the proof of Proposition 4.5 in [GO]). Note that the number of such disks is finite, and hence the number of such leaves is finite. We form a branched surface neighborhood N as below. Let X be an arbitrary 3-cell of $\mathcal{C}$. For each type of normal disks in $\mathcal{L} \cap X$, we take a ball in X of the shape (disk) $\times I$ between the highest disk and the lowest disk of this type. Note that every normal disk type contains at least two disks because of the splitting. Then we take a branched surface neighborhood N' as the union of these balls over all the normal disk types in $\mathcal{L} \cap X$ and over all the 3-cells X. Note that $\partial_h N' \subset \mathcal{L}$. Let U be the open I-bundle $N' - \mathcal{L}$. As in the proof of Proposition 4.5 in [GO], we $\mathcal{L}$-split N' by removing all components of U which are bundles over compact surfaces. Let N be the resulting I-bundle. Let B be the branched surface obtained by collapsing the interval fibers of N to points. (We need to perturb vertical boundary of N so that the branch loci of B is in general position.) Then the obtained branched surface B satisfies the conditions (1),(2),(3) and (5) of the definition of essential branched surfaces by the argument in lines 11–18 in the proof of Proposition 4.5 in [GO]. Hence by Lemma 4.3 and its proof in [GO] there is an essential branched surface B' which is obtained from B by applying a sequence of splittings, and fully carries the lamination $\mathcal{L}$. (However, we give some remarks on the proof of Lemma 4.3 in [GO] right after this proof of Lemma 3.2.) Note that B' intersects every 3-cell in a union of pinched normal disks. We can slightly perturb branch loci of B' to be transverse to the 2-skeleton of the cell decomposition $\mathcal{C}$.

Suppose that a branch locus c is contained in a 3-cell. Then c is contained in two normal disks D_1 and D_2. Let Q_i be the subdisk of D_i bounded by c for $i = 1$ and 2. We can take c so that Q_i contains no branch locus entirely for $i = 1$ and 2. Note that $Q_1 \cap Q_2 = \partial Q_1 = \partial Q_2$ since the components of U, which are I-bundles over compact surfaces, are removed. Then the sphere $Q_1 \cup Q_2$ bounds a ball Z in the 3-cell such that $Z \cap B' = Q_1 \cup Q_2$. We can perform a pinching operation on B' along the ball Z. This eliminates the branch locus c. We repeat such operations until B' has no branch locus entirely contained in a 3-cell.

Suppose that a branch locus intersects in an arc α with a 3-cell so that α has its both endpoints in a 2-cell. Then there are two normal disks P_1 and P_2 containing α, and subarcs of the edges β_1 and β_2 of P_1 and P_2 connecting the two points $\partial\alpha$. The loops $\alpha \cup \beta_1$ and $\alpha \cup \beta_2$ bound disks R_1 and R_2 in P_1 and P_2 respectively. We can take α to be outermost, that is, so that R_i does not contain such a subarc of branch locus entirely for $i = 1$ and 2. If $R_1 = R_2$, then we can split B' along $R_1 = R_2$ to push α out of the 3-cell. If $R_1 \neq R_2$, then the loop $\beta_1 \cup \beta_2$ bounds a disk R in a 2-cell. Then the sphere $R \cup R_1 \cup R_2$ bounds a ball Y in the 3-cell. We can perform a pinching operation on B' along this ball Y to push α out of the 3-cell Repeating such operations, we obtain a branched surface in normal form with respect to $\mathcal{C}$. Note that B' is still essential after such splitting operations and pinching operations. Since the number of the points (the branch loci of B) $\cap$ (union of the 2-cells in $\mathcal{C}$) is finite, we see that the sequence of these procedures terminates in finitely many steps to give a branched surface B_* satisfying the conditions (1)–(5) of the definition of a normal form. $\square$

Remark on the proof of Lemma 4.3 in [GO]. In the second sentence of the second paragraph of the proof of Lemma 4.3 in [GO], it is claimed that $\hat{N}(B)$ intersects T twice. However, the authors of this paper took a considerable time to understand this fact. We will give a detailed proof of this fact in Fact 3 in Appendix B.

In the first sentence of the third paragraph of the proof, we take a sequence of splittings of B whose "inverse limit" is the lamination λ. However, there may not be such a sequence of splittings of B. We need to split and isotope λ so that $\partial_h N(B) \subset \lambda$, and need to extend λ by adding interstitial foliations transverse to the open I-bundle structure in the interstitial open I-bundles disjoint from $\partial_v N(B)$. Then we can take such a sequence of splittings of B.

In the second sentence of the third paragraph of the proof, we take a subset K of

$\mathcal{M}(B)$ representing projective transverse measures on λ. The subset K may consist of a single element, the trivial measure 0 on λ. However, it does not matter to the arguments there.

In the fourth sentence of the third paragraph of the proof, the equation $\cap K_i = K$ is given. However, the authors of this paper cannot tell why this equation holds. This equation leads to the fact that there is a branched surface B_n in the sequence of splittings such that B_n carries none of the tori T_i. In the rest of this remark we give another proof of the fact that there is a finite sequence of splittings giving a Reebless branched surface.

Suppose for a contradiction that for every integer i the branched surface B_i carries the same torus T which is compressible in M. We first show that T is isotopic to a leaf of λ by a fiber preserving isotopy in the I-bundle $N(B)$. Let J be an I-fiber of $N(B)$ intersecting T. Since the union $\cup\lambda$ of the leaves of λ is a closed subset of M, $J' = J \cap (\cup\lambda)$ is a closed subset of J. Hence an arcwise connected component of J' is a point or a subinterval in J. Thus an arcwise connected component of $\cup\lambda$ is a surface or an I-bundle. The torus T is contained in $\cup\lambda = \cap_{i=0}^{\infty} N(B_i)$. If T is contained in a surface component of $\cup\lambda$, then T is equal to the surface, and we are done. If T is contained in an I-bundle component W of $\cup\lambda$, then we can isotope T in W so that $T \subset \partial W$ by a fiber preserving isotopy. Since ∂W is a leaf of λ, it implies that T is isotopic to a leaf of λ in the I-bundle $N(B)$ by a fiber preserving isotopy.

By adding parallel leaves, we can assume that the toral leaf T of λ is contained in $\mathrm{Int}N(B)$. We cut the branched surface neighborhood $N(B)$ along T and collapsing the I-fibres to points, to obtain a branched surface B'. Let D be a compressing disk of T. We isotope D near ∂D keeping that $\partial D \subset T$ so that ∂D is transverse to the branch loci of B', and isotope D fixing ∂D so that D is transverse to B'. Then $B' \cap D$ is a train track τ containing the boundary loop ∂D. Since B' does not admit a monogon, τ does not have a monogon face in D. Hence we can see that τ have a 0-gon face Q in D by an easy calculation on Euler characteristic of D. Set $Q' = Q \cap (M - \mathrm{Int}N(B))$. Then the circle $\partial Q'$ bounds a disk Q'' on $\partial_h N(B)$ because $\partial_h N(B)$ is incompressible. Let $\mathcal{F}$ be the thickening of λ in $N(B)$. By the Reeb stability theorem ([Lemma 2.2, GO]) and by the condition that λ does not have a vanishing cycle, $\mathcal{F}$ contains a product foliation $Q'' \times [0,1]$, where $Q'' \times \{0\} = Q''$ and $Q'' \times \{1\}$ is incident to $\partial_h N(B)$. We split B by deleting the interstitial open I-bundle over $\mathrm{cl}(Q'' \times \{1\} - \partial_h N(B))$. Let B'' be the branched surface obtained by the above splitting. Let Q''' be the disk bounded by $\partial Q'' \times \{1\}$ on the compressing disk D. Then we can retake D by replacing Q''' with $Q'' \times \{1\}$ and isotoping slightly off of the split $N(B'')$. This operation decreases either the number of the components of τ or the number of the branched points of τ. Repeating such operations, we can retake D so that $B_* \cap D = \tau = \partial D$, for some branched surface B_* which is obtained from B by splitting operations with respect to λ. Similar arguments as above show that D intersects $\mathcal{F}$ in parallel circles which bound parallel disks in $\mathcal{F}$. This implies that ∂D is not essential in the torus T, which is a contradiction.

Proof of Theorem 3.1. It is well known that there is a nice regular cell decomposition $\mathcal{C}$ of $E(L)$ induced from the connected link diagram E. (See, for example, [M0] or [We, chapter 2]). That is, let F be the union of disks $S_0 - \mathrm{Int}\ N(L)$ and twisted rectangles, two intersecting in a polar axis in each crossing ball. (Note that F is the union of the white spanning surface and the black spanning surface.) Then F divides the exterior $E(L)$ into two 3-cells. The 1-cells of $\mathcal{C}$ are the polar axes and the arcs of $\partial F \cap N(L)$.

Then there is an essential lamination $\mathcal{L}$ in $E(L)$ which is carried by an essential branched surface in a normal form with respect to $\mathcal{C}$ by Theorem in [B] and Lemma 3.2. It is easy to check that it satisfies the conditions of standard position in section 2 (in fact, the conditions (1)–(5) of the definition of normal branched surface respectively correspond to the conditions (1)–(5) of the definition of standard position).

Moreover, if $\mathcal{L}_0$ is essential in some 3-manifold M obtained by a Dehn filling, then we can take $\mathcal{L}$ to be also essential in M by Addendum to Theorem in [B].

When $\mathcal{L}_0$ is inessential in all 3-manifolds obtained by a Dehn filling along a boundary component T, $\mathcal{L}_0$ contains a toral leaf parallel to T by Theorems 1 and 2 in [W]. Hence, if $\mathcal{L}_0$ does not contain a toral leaf parallel to a boundary component, then $\mathcal{L}_0$ is essential in some 3-manifold M obtained by a Dehn filling. Then by the previous paragraph, the lamination $\mathcal{L}$ is essential in M. Thus $\mathcal{L}$ does not contain a toral leaf parallel to a boundary component of $\partial E(L)$. $\square$

§4 A method for examining the essentiality of branched surfaces

Let L, S, E, $S_\pm$, S_0, $\tau_\pm$ be as in section 2. In this section, we suppose that L is an alternating link, and E is an alternating diagram which is reduced to have no nugatory crossing as in Figure 4.1.

$\boxed{\textit{Figure } 4.1}$

By Menasco [M1], if the link L is non-split and prime, then the diagram E is connected and prime, that is, S contains no embedded circle meeting E twice transversely and bounding no disk intersecting E in a simple arc. He also showed that closed incompressible surfaces in complements of alternating links are isotoped to be in "standard position" with respect to E in [M1], and claimed that closed surfaces in standard position are incompressible under certain conditions in [M2]. C. Delman and R. Roberts constructed essential laminations in the 3-manifolds obtained by non-trivial Dehn surgery on alternating knots in S^3 other than $(2,p)$-torus knots in [DR]. As a corollary they showed that all alternating knots have property P. The essential laminations they constructed meet the attached solid tori of the surgeries. However, it is still unknown whether there exist essential laminations without boundary in alternating knot complements other than $(2,p)$-torus knots which survive all non-trivial Dehn surgeries, i.e., are also essential after all non-trivial Dehn surgeries.

In this section, we study on essential laminations without boundary, and closed branched surfaces in complements of alternating links. We first show that under certain conditions a branched surface in standard position with respect to E satisfies the conditions (1)-(3) of the definition of essential branched surfaces (Theorem 4.1). We note that in Appendix B, we show some (known) methods for proving non-existence of disks of contact and Reeb branched surfaces for general branched surfaces in 3-manifolds. Recall that a train track τ is a graph imbedded in a surface. In this section, we treat train tracks with valency three at each vertex. Note that τ has a certain kind of smooth structure as follows. Let v be a vertex of τ and e, f, g the three edges incident to v. Then two unions of two edges, say $e \cup f$ and $e \cup g$ are smooth arc, and the other union $f \cup g$ is not smooth. We say that v has the *smooth valency* equal to 2 along e, and smooth valencies of v along f and g are both 1.

We return to our situation. Let B be a branched surface in the exterior $E(L)$ of L. Suppose B is in standard position with respect to E with a system of generating disks $D_1^+, \ldots, D_m^+$ ($D_1^-, \ldots, D_n^-$ resp.) for $B \cap B_+$ ($B \cap B_-$ resp.). Let Λ be the union of the branch loci of B.

We say that B is *nice* if it satisfies the six conditions below.

(1) τ_+ and τ_- are both connected.
(2) No smooth circle of the train track τ_+ (τ_- resp.) bounding a disk D_i^+ (D_j^- resp.) of a system of generating disks for $B \cap B_+$ ($B \cap B_-$ resp.) meets the same region of E more than once.
(3) There is no such pattern as shown in Figure 4.2. In Figure 4.2, γ is an arc of $(B - \Lambda) \cap S_0$ in a region R. There is a very small arrow v tangent to R incident and normal to γ. Let α and β be the smooth circles in τ_+ and τ_- respectively such that $\gamma \subset (\alpha \cap \beta)$ where they are the innermost smooth circles containing γ in the direction of v in τ_+ and τ_- respectively. Let R_+ and R_- be intersections

of R and the innermost disks bounded by α and β on S_+ and S_- respectively. There is a common bubble X meeting both R_+ and R_-, and α and β miss the arc $X \cap R$. Let R_1 and R_2 are regions of E adjacent to R around X. The circle α intersects R_1 and the circle β intersects R_2. The arrangement of R, R_1, R_2 and the overstrand at X is as in Figure 4.2 (or its mirror image), that is, the component of $L \cap S_0$ between R and R_1 (resp. R and R_2) connects with the understrand (resp. overstrand) of X.

(4) No region of E contains an edge e of $\tau_+ \cap \tau_-$ such that smooth valencies at the two endpoints of e along e are both 2. See Figure 4.3.

(5) There is no such pattern as shown in Figure 4.4. There, X is a bubble, and R_1, R_2, R_3 are regions of E appearing in this order around X. There are two edges of τ_+, say $\alpha \subset (R_1 \cup X \cup R_2)$, and of τ_-, say $\beta \subset (R_2 \cup X \cup R_3)$ such that $\alpha \cap R_2 = \beta \cap R_2$ and each of α and β meets X just once. Along the edges α and β their endpoints have smooth valencies 2.

(6) There is no such pattern as shown in Figure 4.5. In Figure 4.5, $X_1, \cdots, X_n$ are bubbles, and $R_1, \cdots, R_{2n+1}$ are regions of E possibly $X_i = X_j$ (resp. $R_i = R_j$) for some i and j with $|i - j| \geq 2$. The regions R_{2i-1}, R_{2i} and R_{2i+1} appears in this order around X_i as in Figure 4.5, where $1 \leq i \leq n$. In case n is odd, it is as in Figure 4.5(1). That is, there are edges of τ_+, say $\alpha_1 \subset (R_1 \cup X_1 \cup R_2)$ and $\alpha_i \subset (R_{4i-4} \cup X_{2i-2} \cup R_{4i-3} \cup X_{2i-1} \cup R_{4i-2})$, where $2 \leq i \leq \frac{n+1}{2}$, and of τ_-, say $\beta_j \subset (R_{4j-2} \cup X_{2j-1} \cup R_{4j-1} \cup X_{2j} \cup R_{4j})$, where $1 \leq j \leq \frac{n-1}{2}$, and $\beta_{\frac{n+1}{2}} \subset (R_{2n} \cup X_n \cup R_{2n+1})$ such that $\alpha_i \cap R_{4i-2} = \beta_i \cap R_{4i-2}$ and that $\alpha_{i+1} \cap R_{4i} = \beta_i \cap R_{4i}$. Along the edges α_i and β_j, their endpoints have smooth valencies 2. In case n is even, it is as in Figure 4.5(2). That is, there are edges of τ_+, say $\alpha_1 \subset (R_1 \cup X_1 \cup R_2)$, $\alpha_i \subset (R_{4i-4} \cup X_{2i-2} \cup R_{4i-3} \cup X_{2i-1} \cup R_{4i-2})$, where $2 \leq i \leq \frac{n}{2}$, and $\alpha_{\frac{n+2}{2}} \subset (R_{2n} \cup X_n \cup R_{2n+1})$, and of τ_-, say $\beta_j \subset (R_{4j-2} \cup X_{2j-1} \cup R_{4j-1} \cup X_{2j} \cup R_{4j})$, where $1 \leq j \leq \frac{n}{2}$, such that $\alpha_i \cap R_{4i-2} = \beta_i \cap R_{4i-2}$ and that $\alpha_{i+1} \cap R_{4i} = \beta_i \cap R_{4i}$. Along the edges α_i and β_j, their endpoints have smooth valencies 2. We also admit the patterns where edges of τ_+(resp. τ_-) play the role of τ_-(resp. τ_+) in Figure 4.5.

$\boxed{\textit{Figures } 4.2, 4.3, 4.4 \textit{ and } 4.5}$

Theorem 4.1. *Let L be an alternating link in S^3, S the projection 2-sphere, E a reduced connected prime alternating diagram of L on S. Let B be a branched surface without boundary in standard position with respect to E. Then no component of $\partial_h N(B)$ is* *a sphere, M − Int $N(B)$ is irreducible and ∂M is incompressible in M − Int $N(B)$. Furthermore, if B is nice, then B has no disk of contact, $\partial_h N(B)$ is incompressible in M − Int $N(B)$ and there is no monogon in M − Int $N(B)$. Hence if B is nice, B satisfies the condition (1)-(3) of the definition of essential branched surfaces.*

Before proving Theorem 4.1, we prepare three lemmas which are valid for branched surfaces in standard position with respect to general (not necessarily alternating) diagrams E.

Recall that B is a branched surface in the exterior $E(L)$ of L, where B is in a standard position with respect to E with a system of generating disks $D_1^+, \dots, D_m^+$ ($D_1^-, \dots, D_n^-$ resp.) for $B \cap B_+$ ($B \cap B_-$ resp.). In the following, we suppose that $\partial_h N(B) \cap B_\pm = \partial_h N(B \cap B_\pm)$ and $\partial_v N(B) \cap B_\pm = \partial_v N(B \cap B_\pm)$. Let $E(L \cup B) = \mathrm{cl}(E(L) - N(B))$

Lemma 4.2. *Each component of $E(L \cup B) \cap B_\pm$ is a 3-ball.*

Proof. Since the argument is the same, we prove this only for $E(L \cup B) \cap B_+$. We split $B \cap B_+$ into mutually disjoint smooth disks $D_1^+ \cup \cdots \cup D_m^+$. Each component of the exterior of these disks is a 3-ball. If we pinch these disks with a connected I-bundle such that no component of the resultant branch locus is a closed curve contained in $B \cap B_+$, then the exterior of the obtained branched surface is the union of 3-balls. Since $B \cap B_+$ is obtained by repeating pinchings as above, the lemma follows. □

Lemma 4.3. *$\partial_h N(B) \cap B_\pm$ is a disjoint union of disks.*

Proof. Since the argument is the same, we prove this only for $\partial_h N(B) \cap B_+$. Suppose, for a contradiction, that there is a non-disk component of $\partial_h N(B) \cap B_+$. We take a loop, say C, on the component of $\partial_h N(B) \cap B_+$ which is disjoint from Λ and does not bound a disk on the component. Let $Q_1, \dots, Q_{2m}$ be duplicated parallel copies of $D_1^+, \dots, D_m^+$ in $N(B \cap B_+)$ such that $\partial_h N(B \cap B_+) \subset Q_1 \cup \cdots \cup Q_{2m}$. Let Q_k be the disk such that $Q_k \supset C$. Then C bounds a disk, say D in Q_k. Since C does not bound a disk in $\partial_h N(B \cap B_+)$, we see that there is an annulus component of $\partial_v N(B \cap B_+)$, say A, such that a component of ∂A is contained in C. However this shows that there is a component of Λ contained in D, which contradicts the condition (3) of the definition of standard position. □

Lemma 4.4. *If there is a disk D properly embedded in $E(L \cup B) \cap B_+$ ($E(L \cup B) \cap B_-$ resp.) disjoint from $\partial_v N(B)$ such that ∂D is a union of two subarcs say $\xi_1 \subset \partial_h N(B) \cap B_+$ ($\xi_1 \subset \partial_h N(B) \cap B_-$ resp.) and $\xi_2 \subset (S_+ - L)$ ($\xi_2 \subset (S_- - L)$ resp.). Then we can move $D_1^+ \cup \cdots \cup D_m^+$ ($D_1^- \cup \cdots \cup D_n^-$ resp.) by an isotopy in $N(B \cap B_+)$ ($N(B \cap B_-)$*

resp.) so that there is a component of $D_1^+ \cup \cdots \cup D_m^+$ *(*$D_1^- \cup \cdots \cup D_n^-$ *resp.) , say* D_i^+ *(*D_j^- *resp.) such that* $\xi_1 \subset D_i^+$ *(*$\xi_1 \subset D_j^-$ *resp.).*

Proof. Since the argument is the same, we prove this only for $E(L \cup B) \cap B_+$. Let S be the component of $\partial_h N(B \cap B_+)$ which contains ξ_1. Since $B \cap B_+$ fully carries $D_1^+ \cup \cdots \cup D_m^+$, we can move $D_1^+ \cup \cdots \cup D_m^+$ by an isotopy in $N(B \cap B_+)$ so that $S \subset (D_1^+ \cup \cdots \cup D_m^+)$. Let D_i^+ be the component of $D_1^+ \cup \cdots \cup D_m^+$ such that $D_i^+ \supset S$. This gives the conclusion of Lemma 4.4. $\square$

The proof of Theorem 4.1 is a consequence of the following four lemmas. Hereafter we moreover suppose that L is an alternating link, and E is a reduced, connected alternating diagram.

An imbedded closed surface $F \subset S^3 - L$ is called *pairwise compressible* if there is a disk $D \subset S^3$ meeting L transversely in one point, with $D \cap F = \partial D$ as defined in [M1]. Otherwise we say that F is *pairwise incompressible.*

Lemma 4.5. *$E(L \cup B)$ is irreducible and $\partial E(L)$ is incompressible in $E(L \cup B)$.*

Proof. Suppose, for a contradiction, that there is a compressing disk for $\partial E(L)$ in $E(L \cup B)$. Then it follows either that L is the trivial knot or that L has at least 2-components and contains a component bounding a disk. In the latter case, L is split. On the other hand, since the alternating diagram E is reduced and connected, L is not the trivial knot by, for example, [Ba] and L is non-split in S^3 by [M1]. Hence in both cases we have a contradiction. Thus $\partial E(L)$ is incompressible in $E(L \cup B)$.

Suppose, for a contradiction, that $E(L \cup B)$ is reducible. Let Q be a sphere which does not bound a ball in $E(L \cup B)$. Then we will show that we can retake the splitting sphere Q so that Q is in standard position as in [M1, Proof of Lemma 1], that is, so that Q satisfies the conditions (1)-(4) below.

(1) Q intersects S_+ and S_- transversely and Q intersects each crossing ball bounded by a bubble in zero or several "saddle-shaped" disks as shown in Figure 2.2.

For a splitting sphere Q satisfying the above condition (1), we define the *complexity* $c(Q)$ of Q to be the lexicographically ordered pair (s, t), where s is the number of saddles of Q and t is the sum of the number of components of $Q \cap S_+$ and $Q \cap S_-$. From now on we assume that Q satisfies the above condition and has minimal complexity among all splitting spheres.

Claim. *Q satisfies the following conditions;*

(2) *Each circle of $Q \cap S_+$ (resp. $Q \cap S_-$) bounds a smooth disk in $Q \cap B_+$ (resp. $Q \cap B_-$).*

(3) *Every circle of* $Q \cap S_+$ *and* $Q \cap S_-$ *meets a bubble.*

(4) *No circle of* $Q \cap S_+$ *and* $Q \cap S_-$ *meets the same side of a bubble more than once.*

Proof. From Lemma 4.2, every component of $E(L \cup B) \cap B_\pm$ is a 3-ball. Hence incompressible surfaces properly imbedded in $E(L \cup B) \cap B_\pm$ whose non-empty boundaries are contained in $\partial B_\pm$ are disks.

For (2), suppose not. Then $Q \cap B_\pm$ is compressible in $E(L \cap B) \cap B_\pm$. Let D' be a compressing disk. A surgery of Q with D' produces two spheres, one of them is a new splitting sphere with fewer complexity. This is a contradiction to the minimality of the complexity $c(Q)$.

For (3), suppose not. Then there is a circle which is contained in a region and bounds a subdisk D' of S_0 in the region. If B meets D', then we can find a 0-gon or monogon in D' which contradicts Lemma 2.1. Hence B is disjoint from D'. Then a surgery of Q along D' and a slight isotopy yields a new splitting sphere with fewer complexity, which is a contradiction.

For (4), suppose not. Let C be a circle of, say, $Q \cap S_+$ which meets the same side of a bubble X more than once. We assume C is innermost among such circles on S_+. Since C is innermost, we can take saddles s_1 and s_2 in X such that C meets s_1 and s_2 successively and s_1 and s_2 are adjacent among saddles of Q, that is, there is no saddle of Q between s_1 and s_2 in X. Let d be the disk bounded by C on S_+ such that d contains the subdisk of $X_+ = X \cap S_+$ between the arcs $s_1 \cap X_+$ and $s_2 \cap X_+$.

Suppose, for a contradiction, that there is a saddle, say s', of B between s_1 and s_2. Let α be the boundary of a disk D_i^+ which intersects X in an arc of $X \cap s'$. Then by the condition (4-1) of the definition of standard position of branched surfaces, α is a smooth loop imbedded in τ_+. Since α is contained in d, α meets the same side of X more than once, which contradicts the condition (4-3) of the definition of standard position. Hence there is no saddle of B between s_1 and s_2.

Then as in the proof of [M1, Lemma 1 (ii)], we can isotope Q so as to reduce $c(Q)$, which is a contradiction. $\square$

In general, let F be a surface in $E(L)$ in standard position. Let α be a circle of $F \cap S_+$(resp. $F \cap S_-$) which meets some bubble X. We define the *mate to* α *at* X to be a component of $F \cap S_+$(resp. $F \cap S_-$) which meets the other side of X and contain the subarc of the boundary of the saddle which is incident to α at X.

Every circle of $F \cap S_\pm$ satisfies the following *alternating property* ([M1]);

(*) If a circle $C \subset F \cap S_\pm$ meets two bubbles B_1 and B_2 (they are possibly the same bubble) in succession, then two arcs of $L \cap S_\pm$ in B_1 and B_2 lie on opposite sides of C.

Proof of Lemma 4.5(continued). Let C be a circle of $Q \cap S_\pm$ which is innermost on $S_\pm$. The circle C intersects a bubble by (3) of the above Claim. Then by the alternating property (*) above, we can show that C meets the bubble more than once, otherwise, in the innermost disk we can find another circle which is a mate to C at the bubble, a contradiction. Since Q is in standard position, C meets the distinct sides of the bubble. Then, as in [M1, Proof of Lemma 1], we can show that Q is pairwise compressible. But since S^3 does not contain a non-separating 2-sphere, Q is pairwise incompressible, a contradiction. This completes the proof of Lemma 4.5. □

Lemma 4.6. *No component of $\partial_h N(B)$ is a sphere.*

Proof. Suppose for a contradiction that $\partial_h N(B)$ contains a sphere component Q. From Lemma 4.3, $\partial N_h(B) \cap B_\pm$ is disjoint union of disks. Hence $Q \cap B_\pm$ consists of properly imbedded disks. Then as in the last paragraph of the proof of Lemma 4.5, we can find a disk D of $Q \cap B_\pm$ which meets a bubble more than once. If ∂D meets the same side of the bubble more than once, then, since $Q \subset \partial_h N(B)$, we can find a boundary of D_i^+ or D_j^- violating the condition (4-3) of the definition of standard position, which is a contradiction. If ∂D meets the distinct sides, then as in the last paragraph of the proof of Lemma 4.5 we have a contradiction. □

Lemma 4.7. *Suppose B satisfies the conditions (1), (2) and (3) in the definition of nice branched surface. Then $\partial_h N(B)$ is incompressible in $E(L \cup B)$ and there is no monogon in $E(L \cup B)$.*

Proof. Suppose, for a contradiction, that $\partial_h N(B)$ is compressible in $E(L \cup B)$ or there is a monogon. Let D be a compressing disk of $\partial_h N(B)$ or a monogon. From Lemma 4.3, $\partial_h N(B) \cap B_\pm$ is a disjoint union of disks. Hence, if D is a compressing disk, D intersects $S_\pm$. If D is a monogon and disjoint from $S_\pm$, we can find a pinching between subsurfaces of D_i^+ or D_i^- of the systems of generating disks. This contradicts the condition (4-1) of the definition of standard position of branched surfaces. Hence D intersects $S_\pm$.

By replacing D if necessary, we will show that we can put D in standard position as in [M2, Lemma 4], that is, so that D satisfies the conditions (1)-(5) below.

(1) D intersects S_+ and S_- transversely and D intersects each crossing ball bounded by a bubble in zero or several "saddle-shaped" disks as shown in Figure 2.2.

For a compressing disk or monogon D satisfying the above conditions, we define the *complexity* $c(D)$ of D to be the lexicographically ordered pair (s, t), where s is the number of saddles of D and t is the sum of the number of components of $D \cap S_+$ and $D \cap S_-$.

From now on we assume that D has minimal complexity among all compressing disks or monogons.

Claim 1. *D satisfies the following conditions;*

(2) *Each circle of $D \cap S_+$ ($D \cap S_-$ resp.) bounds a smooth disk in $D \cap B_+$ ($D \cap B_-$ resp.).*
(3) *Every circle of $D \cap S_+$ and $D \cap S_-$ meets a bubble.*
(4) *No circle of $D \cap S_+$ and $D \cap S_-$ meets the same side of a bubble more than once.*
(5) *No arc of $D \cap S_+$ and $D \cap S_-$ contains a subarc contained in a region of E whose endpoints are contained in a bubble.*

Proof. For (2), (3) and (4), the same argument in the proof of Claim in the proof of Lemma 4.5 will do. For (5), the argument in the proof of (4) of Claim in the proof of Lemma 4.5 will do. □

Claim 2. *There is no circle in $D \cap S_\pm$.*

Proof. Suppose there is a circle in $D \cap S_\pm$, say in $D \cap S_+$. Take an innermost circle $C \subset D \cap S_+$ on S_+. Let d_C be the innermost disk bounded by C on S_+. By the alternating property (*), either C meets a bubble more than once or we can find a mate to C in d_C. In the former case, since D is in standard position, C meets the distinct sides of the bubble. Then, as in the last paragraph of the proof of Lemma 4.5, we have a contradiction. Hence we can find a mate to C in d_C. Since C is an innermost circle, the mate is an arc and connects to τ_+. Next take an innermost circle C' of $D \cap S_+$ in $S_+ - \mathrm{Int}\, d_C$ bounding an innermost disk $d_{C'}$ in $S_+ - \mathrm{Int}\, d_C$. (Possibly $C' = C$.) Then by the same argument above, we can find a component of τ_+ in $d_{C'}$. Hence τ_+ is disconnected, which contradicts the condition (1) in the definition of nice branched surface. Therefore there is no circle in $D \cap S_\pm$. □

Now we form a graph G on D such that the "square" vertices of G are saddles in D and the edges of G are arcs of $D \cap S_0$. Note that every vertex has valency equal to 4. An edge e is an *outermost arc* if it is incident to no vertex and cuts off a subdisk d' from D such that $d' \cap G = e$ and that d' is disjoint from the vertex of D in case D is a monogon.

Claim 3. *There is no outermost arc in G.*

Proof. Suppose G contains an outermost arc e. Then e is contained in a region, say R, of E. Without loss of generality, we assume $d' \subset B_+$. Then, by Lemma 4.4, the endpoints a and b of e connects with a smooth circle α contained in τ_+ which is the boundary of a disk D_i^+ for some i. We take α so that α is innermost among such circles on the side of e on S_+.

Suppose one of the two arcs α_1 of $\alpha - (a \cup b)$ is contained in the region R. Then, by Lemma 2.1, and since α is innermost, $e \cup \alpha_1$ bounds a disk d in R such that $(\text{Int } d) \cap B = \emptyset$. If $(\text{Int } d) \cap D \neq \emptyset$, then, by (3) in Claim 1, D meets d in arcs. Then we replace e with an outermost arc of $d \cap D$ and use d to denote the new disk cobounded by the outermost arc and a subarc of α_1. By performing surgery of D along d, we obtain two disks, one of which must be a compressing disk or a monogon. We call this new disk D'. Then the sum of the number of the components of $D' \cap S_+$ and $D' \cap S_-$ is less than that of $D \cap S_+$ and $D \cap S_-$, which contradicts the minimality of the complexity of D.

Therefore both of two arcs of $\alpha - (a \cup b)$ go out of R. Since $\alpha = \partial D_i^+$, α violates the condition (2) in the definition of nice branched surfaces. Hence there is no such outermost arc. $\square$

A *face* of G is the closure of a component of $D - G$. Now we define an *outermost fork*, which is a subgraph of G as shown in Figure 4.6. That is, there are two adjacent faces D_+ and D_- of G, two arcs $\eta_+ \subset D \cap S_+$ and $\eta_- \subset D \cap S_-$ and a saddle s such that ∂D_+(resp. ∂D_-) consists of η_+(resp. η_-) and a subarc of ∂D and η_+ and η_- meets a common saddle s and no other saddles.

$\boxed{Figure\ 4.6}$

Proof of Lemma 4.7(continued). From Claim 2, there is no circle in $D \cap S_\pm$ which implies that each component of G is simply connected, i.e., a tree. Since there is no outermost disk by Claim 3, by an outermost fork argument, we can find two outermost forks, in one of them its two faces D_+ and D_- are disjoint from the vertex of D in case D is a monogon. Let η_+, η_- and s be as in the definition of outermost fork. Let X be the bubble containing s. By Lemma 4.4, around $\eta_+ \cup X \cup \eta_-$, there are two smooth circles $\alpha \subset \tau_+$ and $\beta \subset \tau_-$ which bound smooth disks D_i^+ and D_j^- for some i and j such that D_i^+ and D_j^- contains $\partial D_+ - \eta_+$ and $\partial D_- - \eta_-$, respectively. Let d_+(resp. d_-) be the smooth disk on S_+ (resp. S_-) bounded by α (resp. β) on the side of η_+ (resp. η_-). We assume that α (resp. β) is innermost among such circles bounding the innermost disk d_+ (resp. d_-). Let R be the region of E containing the point $\partial\eta_+ \cap \partial\eta_-$, and R_1(resp. R_2) containing $\partial\eta_+ - \partial\eta_-$(resp. $\partial\eta_- - \partial\eta_+$). Let γ be the arc of $(B - \Lambda) \cap S_0$ containing $\partial\eta_+ \cap \partial\eta_-$. See Figure 4.7.

$\boxed{Figure\ 4.7}$

Suppose, for a contradiction, that α or β meets the arc $X \cap R$, say α does. Since, by the condition (2) of the definition of nice branched surface, α meets R exactly once, then there is a subarc $\xi \subset \alpha$ connecting $\partial\eta_+ \cap \partial\eta_-$ and X such that Int ξ does not meet bubbles. Then ξ, $\eta_+ \cap R$ and a subarc of $X \cap R$ form a loop which bounds a disk d. See Figure 4.8.

$\boxed{\textit{Figures } 4.8 \textit{ and } 4.9}$

Claim 4. *$($Int $d) \cap B = \emptyset$.*

Proof. Suppose, for a contradiction, that (Int $d) \cap B \neq \emptyset$. Let δ be the boundary of a disk, say D_k^+, which contains a part of $B \cap S_+$ contained in Int d. By the condition (4-2) of the definition of standard position, δ goes out of d. There are three cases as in Figure 4.9; (1)δ meets X twice, (2)δ meets the point $\partial\eta_+ \cap \partial\eta_-$ twice, or (3)δ meets X and $\partial\eta_+ \cap \partial\eta_-$. For (1), δ violates the condition (4-3) of the definition of the standard position. For (2), it contradicts (4-1) of the definition of the standard position. For (3), it contradicts the way of choice of α. Hence we have a contradiction. □

Claim 5. *There is no pattern as in Figure 4.8 such that $($Int $d) \cap B = \emptyset$.*

This claim corresponds to [S, Lemma 4.18], but we include the proof for the convenience for the reader.

Proof. First suppose (Int $d) \cap D \neq \emptyset$. Then by (3), (4) and (5) of Claim 1 and Claim 3, (Int $d) \cap D$ consists of arcs connecting $\xi \cap d$ and $X \cap R$. Then in d we can find another pattern as in Figure 4.8. By replacing the pattern with the innermost one in d, we assume that (Int $d) \cap D = \emptyset$.

Let s' be the saddle contained in X and incident to ξ. Since (Int $d) \cap (D \cup B) = \emptyset$, s and s' are adjacent saddles in X. Now we take a look at the other side of the saddles s and s'. Then there is a subarc of τ_+, say ξ', and an arc of $D \cap S_+$, say η', which are mates of ξ and η_+ at X respectively.

First we consider the special case, where η' connects with ξ' such that subarcs of η', ξ' and $X \cap R_2$ together cobounds a disk d' in R_2 and that two points $\xi \cap \eta_+$ and $\xi' \cap \eta'$ is connected by a subarc of ∂D, say ζ. Moreover we assume that d' is innermost among such disks. See Figure 4.10. Note that, since d' is contained in R_2, Int $(\xi' \cap d')$ and Int $(\eta' \cap d')$ does not meet bubbles. Moreover, by (3) of Claim 1, Claim 3 and Claim 4, and the fact that s and s' are adjacent saddles in X, (Int $d') \cap (D \cup B) = \emptyset$. We take an arc δ on $B \cap B_-$ which is parallel to one of an edge of s' as in Figure 4.11. Then the loop consisting of a subarc of ξ, ζ, a subarc of ξ' and δ bounds a disk Q in $B \cap B_-$. We isotope D along Q so as to eliminate the saddle s as shown in Figure 4.12. This isotopy reduces $c(D)$.

$\boxed{\textit{Figures } 4.10,\ 4.11,\ 4.12 \textit{ and } 4.13}$

In general case, we can also apply the above argument as follows. Take a triangle xyz on S_- as shown in Figure 4.13, where $x = \eta_+ \cap \xi \cap d$, $y = \eta' \cap X \cap R_2$ and $z = \xi' \cap X \cap R_2$. We slightly isotope this triangle into B_- so that $x \in \partial D \cap (\text{Int } B_-)$, $y \in \eta' \cap (\text{Int } R_2)$ and $z \in \xi' \cap (\text{Int } R_2)$. See Figure 4.14. We isotope D along this triangle so that $\tau_\pm$, $D \cap S_\pm$ and ∂D is as in Figure 4.15. Then we isotope D so as to eliminate the saddle s, which reduce $c(D)$. $\square$

$\boxed{\textit{Figures } 4.14 \textit{ and } 4.15}$

Hence, by Claims 4 and 5, we have shown that α and β miss $X \cap R$. Thus we can find the pattern as in Figure 4.2, which contradicts (3) of the definition of nice branched surfaces. This completes the proof of Lemma 4.7. $\square$

Hereafter we moreover suppose that E is a prime diagram.

Lemma 4.8. *Suppose B satisfies the conditions (4), (5) and (6) in the definition of nice branched surface. Then B has no disk of contact.*

Proof. Suppose, for a contradiction, that $N(B)$ contains a disk of contact D. We assume that $N(B)$ meets $S_\pm$ in a union of I-fibers. By the definition, D meets fibers of $N(B)$ transversely. Since ∂D is contained in $\partial_v N(B)$, from the condition (3) of the definition of standard position of branched surfaces, D meets $S_\pm$. Hence D satisfies the following condition;

(1) D intersects S_+ and S_- transversely and intersects each crossing ball bounded by a bubble in zero or several "saddle-shaped" disks as shown in Figure 2.2.

Now we form a graph G on D as in the proof of Lemma 4.7. Note that here G may be disconnected and contain non-disk faces.

Claim 1. *G does not contain an arc component disjoint from saddles.*

Proof. Suppose, for a contradiction, that G contains an arc e disjoint from saddles. Since e does not meet bubbles, e is contained in a region of E. If e meets a fiber of $N(B)$ more than once, we can find in the region a smooth circle imbedded in $\tau_\pm$ or a monogon, which contradicts Lemma 2.1. Hence e does not meet a fiber of $N(B)$ more than once.

Then we can find a pattern as in Figure 4.3, which contradicts the condition (4) of the definition of nice branched surfaces. Hence G does not contain such an arc. $\square$

We take a component G' of G such that G' meets ∂D and we regard G' as a graph on D. Since G' is connected, every face of G' is a disk.

Claim 2. *There is no outermost fork in G'.*

Proof. Suppose there is an outermost fork. If two arcs η_+ and η_- of the outermost fork meet a fiber of $N(B)$ more than once, then B violates the conclusion of Lemma 2.1. Hence η_+ and η_- meet every fiber of $N(B)$ at most once. Thus we can find a pattern as in Figure 4.4, which contradicts the condition (5) of the definition of a nice branched surfaces. $\square$

Claim 3. *G' contains a pattern depicted in Figure 4.16, that is, for $n \geq 2$, there are $n+1$ faces $D_1, \cdots, D_{n+1}$ of G', arcs $\gamma_1, \cdots, \gamma_{n+1} \subset D \cap S_\pm$ and saddles $s_1, \cdots, s_n$ such that ∂D_i is a union of γ_i and a subarc of ∂D. Moreover γ_i meets two saddles s_{i-1} and s_i for $2 \leq i \leq n$, and γ_1 (γ_{n+1} resp.) meets only one saddle s_1 (s_n resp.).*

$\boxed{\textit{Figure } 4.16}$

Proof. In this proof, we regard every point of $\partial D \cap S_0$ as also a vertex of G' and it is called a *boundary vertex*. Other vertices originating from saddles are called *inner vertices*. We also regard a subarc of ∂D connecting two adjacent boundary vertices as an edge of G' and call it a *boundary edge*. Other edges originating from $D \cap S_0$ are called *inner edges*. Let f_i denote the number of i-gons of G', that is, a face of G' with i edges. We assign the number $i - 4$ for each i-gon in G'. Then the following equality holds.

Subclaim 1. $\sum_i (i-4)f_i = -4$

Proof. Let f, e and v denote the numbers of faces, edges and vertices of G' respectively. Let v_∂, v_i, e_∂ and e_i denote the numbers of the boundary vertices, inner vertices, boundary edges and inner edges, respectively.

We have

(1) $f = \sum_i f_i$, $v = v_\partial + v_i$, $e = e_\partial + e_i$ and $v_\partial = e_\partial$,
(2) $v - e + f = 1$ (Euler's formula),
(3) $2e = 3v_\partial + 4v_i$ (by the valencies of the vertices), and
(4) $2e = \sum_i if_i + e_\partial$ (by the number of the edges of the faces).

From (1), (2) and (3), we have $2f = \sum_i 2f_i = v_\partial + 2v_i + 2$. From (1), (3) and (4), we have $\sum if_i = 2v_\partial + 4v_i$. From the above two equations, we obtain the equation in Subclaim 1. □

A face of G' is called an *inner face* if it meets only inner edges. A face which meets a boundary edge is called a *boundary face.*

Subclaim 2. *Every inner face has even vertices.*

Proof. Otherwise we can find a circle on S meeting E in odd points, which is a contradiction. □

Suppose there is an inner 2-gonal face. Let C be the loop of $D \cap S_\pm$ which is the boundary of the 2-gonal face. If C meets two distinct bubbles, then, from the alternating property(*), we can show that E is composite. See Figure 4.17. Thus C meets a bubble, say X, twice. Then C meets the same side of X twice, otherwise we can form a loop on $S_\pm$ meeting E exactly once, which is a contradiction. Let ξ be one of the two components of $C - X$. Then the interior of ξ does not meet bubbles and ξ and a subarc of $X \cap S_0$ cobounds a disk d whose interior is contained in a region. We consider ξ is carried by $\tau_\pm$. If there is a smooth arc imbedded in $\tau_\pm \cap d$ which meets Int d and cobounds a disk together with a subarc of $\partial d \cap X$ as in Figure 4.18, then we take an outermost such arc, replace ξ and d with the outermost arc and its outermost disk and also call them ξ and d. If $\tau_\pm$ meets Int d, then we can find a 0-gon or monogon contained in a region, which contradicts Lemma 2.1. Hence (Int $d) \cap \tau_\pm = \emptyset$. Then we can find the boundary of a disk $D_i^\pm$ which meets the same side of X more than once. It contradicts the condition (4-3) of the definition of standard position of B. Hence there is no inner 2-gonal face. This together with Subclaim 2 shows that for every inner i-gonal face, $i - 4 \geq 0$.

Figures 4.17 *and* 4.18

Note that every boundary face meets at least three vertices. Hence, for only boundary 3-gons, $i - 4 < 0$, and there are at least four 3-gons by Subclaim 1. By Claim 2, there is no outermost fork in G'. Suppose that G' does not contain a part as in Figure 4.10. Then it follows that among the boundary faces between every pair of boundary 3-gons, there is a face which is not a 4-gon. Then it follows that $\sum(i-4)f_i \geq 0$, which contradicts Subclaim 1. This completes the proof of Claim 3. □

Proof of Lemma 4.8(continued). By Claim 3, G' contains a part as in Figure 4.16. By the same argument as in the proof of Claim 1 and in the paragraph right after the proof of Subclaim 2, each arc γ_i meets a fiber of $N(B)$ at most once and each pair of saddles s_i and s_{i+1} is contained in distinct bubbles X_i and X_{i+1} respectively. Then we can find a pattern as in Figure 4.5, which contradicts the condition (6) of the definition of nice branched surfaces. This completes the proof of Lemma 4.8. $\square$

Proof of Theorem 4.1. Theorem 4.1 follows from Lemmas 4.5, 4.6, 4.7 and 4.8. $\square$

§5 Existence of locally affine laminations carried by branched surfaces

Let L, S, E, $S_\pm$, $B_\pm$, and S_0 be as in section 2. Let B be a branched surface in standard position with respect to the diagram E. In this section, we give some conditions for B to fully carry a lamination by using admissible weights on some train track obtained from $\tau_\pm$. In Theorem 5.3, under a technical condition on B, we give a necessary and sufficient condition for B to fully carry a lamination which is affine (for the definition, see below) as a lamination in $N(B)$. In Theorem 5.4, we consider the general setting. We give a necessary condition for B to fully carry a lamination which is affine as a lamination in $N(B)$, and show that under this condition B fully carries an affine lamination after adequate splitting operations.

For the statement of the result, we first prepare some terminologies.

In general, let τ be a train track embedded in a surface F, and m the number of edges of τ. The *switches* of τ are the vertices of the graph τ. We fix an ordering on the edges $e_1, \dots, e_m$ arbitrarily. An m-tuple of non-negative real numbers $\mathbf{w} = (w_1, \dots, w_m)$ is a *system of admissible weights* on τ if the switch condition is satisfied at each vertex of τ (see Figure 5.1). That is, if e_i, e_j and e_k are incident to a vertex v with v having the smooth valency 2 along e_i (see section 4 for the definition of smooth valency), then $w_i = w_j + w_k$. We often regard $\mathbf{w}$ as an element of m-dimensional real vector space $\mathbb{R}^m$.

We say that $\mathbf{w}$ is *positive* if each w_i is a positive number.

$$\boxed{Figures\ 5.1, 5.2\ and\ 5.3}$$

For a train track τ, there is a fibered neighborhood $N(\tau)$ in F locally modelled as in Figure 5.2.

For a positive system of admissible weights $\mathbf{w} = (w_1, \dots, w_m)$, we can construct a measured neighborhood $N_{\mathbf{w}}(\tau)$ with a measured foliation $(\mathcal{F}, \mu)$, where μ is a transverse measure of $\mathcal{F}$ invariant under the translation along the leaves of $\mathcal{F}$, and is obtained from $\mathbf{w}$. (For a detailed discussion on the transverse invariant measures, we refer to [FLP].) See Figure 5.3. Note that $\mathcal{F}$ has a finite number of singular leaves, where the singularities correspond to the vertices of τ. If each entry of $(w_1, \dots, w_m)$ is a non-negative integer, then this gives also a union of mutually disjoint simple closed curves $\mathcal{L}$ in $N(\tau)$ with the counting measure μ_c, that is, μ_c is the measure such that for a simple closed curve ℓ on F in general position with respect to $\mathcal{L}$, $\mu_c(\ell)$ is the number of the points of $\ell \cap \mathcal{L}$. Then we say that $(w_1, \ldots, w_m)$ *represents* the simple closed curves.

Let M be a compact 3-manifold, and B a branched surface in M. Note that each 2-manifold carried by B is properly embedded in M. The *sectors* of B are the metric completions of the components of $B-$(the branch loci). Let $S_1, \ldots, S_n$ be the sectors of B. Then we can assign a non-negative real numbers w_i, called a *weight*, to each sector S_i. We say that a system of weights on the sectors $(w_1, \ldots, w_n)$ is *admissible* if it satisfies the following *switch condition* at each branch locus.

> Recall that the branch loci of B is an immersed 1-manifold with finitely many transverse self intersection. Then we remove the intersection points from the branch loci to obtain a system of mutually disjoint 1-manifolds in M. Let ρ be one of them, and p a point in Int (ρ). Then there is a regular neighborhood D_p of p such that $D_p \cap \rho$ is an arc properly embedded in D_p and that $B \cap D_p$ consists of three half-disks, say Δ_1, Δ_2, Δ_3, with sharing $D_p \cap \rho$ as their diameters. Here we may suppose that $\Delta_1 \cup \Delta_2$ and $\Delta_1 \cup \Delta_3$ are smooth disks. Let S_i, S_j, S_k be the sectors which contains Δ_1, Δ_2, Δ_3 respectively. (Note that two or three of S_i, S_j, S_k might coincide.)
>
> Then we have
>
> $$w_i = w_j + w_k.$$

Considering all the circles and subarcs of the branch loci as above, we obtain the *system of the switch equation* for B. We say that $(w_1, \ldots, w_n)$ is *positive* if each w_i is a positive number. For a positive system of admissible weights $\mathbf{w} = (w_1, \ldots, w_n)$, we can construct a measured neighborhood $N_{\mathbf{w}}(B)$ with a measured foliation $(\mathcal{F}, \mu)$, where μ is a transverse measure of $\mathcal{F}$ invariant under the translation along the leaves of $\mathcal{F}$, and is obtained from $\mathbf{w}$. See Figure 5.4.

$\boxed{\textit{Figure } 5.4}$

Note that $\mathcal{F}$ has a finite number of singular leaves, where the singularities correspond to the branch loci of B.

Note that if each entry of $(w_1, \ldots, w_n)$ is a non-negative integers, then this gives a union of mutually disjoint surfaces in $N(B)$ with counting measure μ_c. Then we say that $(w_1, \ldots, w_n)$ *represents* the surface. Obviously there is a 1 to 1 correspondence between the set of admissible integral weights and the set of the fiber preserving isotopy classes of unions of mutually disjoint compact surfaces carried by B and properly embedded in M. (Note that surfaces carried by incompressible branched surfaces are incompressible. See Theorem 1 of [FO] and Theorem 2 of [O'].)

We return to our situation.

For the proof of the next proposition, see Appendix C.

Proposition 5.1. *Let L be a link with a diagram E, and B a closed branched surface in standard position with respect to E. Let $\mathcal{L}_\pm$ be a lamination fully carried by the branched surface $B \cap B_\pm$, which is a pinching of a system of generating disks by the definition of standard position. Then there is another system of generating disks $E_1, \dots, E_p$ for $B \cap B_\pm$ such that each leaf of $\mathcal{L}_\pm$ is isotopic to some E_i in the I-bundle $N(B \cap B_\pm)$ by a fiber preserving isotopy. For each E_i, the union of the leaves of $\mathcal{L}_\pm$ which are isotopic to E_i by fiber preserving isotopies is a closed subset of $B_\pm$.*

Note that $\mathcal{L}_\pm$ may be a lamination without an affine structure in the above proposition.

Here we prove:

Lemma 5.2. *There exist only finitely many systems of generating disks for $B \cap B_\pm$, and they are constructible.*

Proof. Since the argument is the same, we prove this lemma only for $B \cap B_+$. We first describe a method for obtaining all systems of generating disks for $B \cap B_+$.

If $B \cap B_+$ is a disjoint union of disks, then the system of the components of $B \cap B_+$ gives a unique system of generating disks, and we are done. Suppose that a component of $B \cap B_+$ is not a disk. Then we first take a properly embedded smooth disk in B_+, say D_1, which is contained in $B \cap B_+$, and is outermost in B_+, i.e., there exists a component H of $B_+ - D_1$ such that $H \cap B = \emptyset$. It is clear that there are only finitely many choices of D_1. Let $\mathcal{S}$ be a union of sectors of $B \cap B_+$. We say that $\mathcal{S}$ is *admissible* with respect to D_1 if $\mathcal{S} \subset D_1$, and $\mathrm{cl}(B \cap B_+ - \mathcal{S})$ is a branched surface. Since $B \cap B_+$ has only finitely many sectors, we see that there exist only finitely many unions of sectors which are admissible with respect to D_1. Then let B_1 be one of the branched surfaces obtained from $B \cap B_+$ by removing a union of sectors which is admissible with respect to D_1.

Then we apply the above arguments to B_1, and so on. We note that if there does not exist an outermost disk in B_1, then we leave B_1 out of consideration. Since the number of the sectors of $B \cap B_+$ is finite, we see that all of these procedures terminate in finitely many steps to obtain finitely many systems of mutually disjoint disks properly embedded in B_+.

We claim that any system of generating disks for $B \cap B_+$ can be obtained by a procedure as above. In fact, any system of generating disks for $B \cap B_+$ has an outermost disk that can be regarded as D_1 above, and we can set $\mathcal{S}$ to be the union of sectors of D_1 disjoint from the other disks of the system.

Since there are finitely many choices of D_1 and $\mathcal{S}$ in every step, there are finitely many systems of generating disks for $B \cap B_+$. $\square$

Example. We will consider $B \cap B_+$ which is a union of two smooth discs D_1 and D_2 properly embedded in B_+ as below. The subsurface of pinching $D_1 \cap D_2$ is a rectangle R such that the union of a pair of two opposite edges of R are exactly $\partial D_1 \cap \partial D_2$ and that the other two edges are the branch loci of the branched surface $D_1 \cup D_2$. Note that the branch loci are properly embedded in D_1 and D_2.

Let Γ_1 and Γ_2 (Δ_1 and Δ_2 resp.) be the closures of the components of $D_1 - R$ ($D_2 - R$ resp.) such that $\Gamma_i \cap \Delta_i$ ($i = 1, 2$) is a component of the branch arcs. Then the branched surface $D_1 \cup D_2$ can be regarded as the union of the three smooth disks D_1, $\Gamma_1 \cup R \cup \Delta_2$, D_2, or of the three smooth disks D_1, $\Gamma_2 \cup R \cup \Delta_1$, D_2. This shows that there are three systems of generating disks for $B \cap B_+$, and by the argument as in the proof of Lemma 5.2 we can show that these are all of the possible systems.

In general, as defined in [O], a *transverse affine structure* for a lamination $\mathcal{L}$ embedded in a 3-manifold M is a transverse invariant measure μ for the preimage $\tilde{\mathcal{L}}$ of $\mathcal{L}$ in the universal cover $\tilde{M}$ of M such that there exists a homomorphism $\phi : \pi_1(M) \to \mathbb{R}_+$ which satisfies the condition below.

> For each $\alpha \in \pi_1(M)$, we have $\alpha^*(\mu) = \phi(\alpha) \cdot \mu$, where $\alpha^*(\mu)$ is the pull-back of the measure μ with α regarded as a covering translation.

The lamination $\mathcal{L}$ together with the transverse affine structure μ is called an *affine lamination*.

For any positive system of admissible weights on a branched surface, it is known that there is a measured lamination corresponding to the weights. For a proof of this, see Theorem 2.1 in Chapter II of Morgan-Shalen's paper [MS]. Here we give another construction of such a lamination, which must be well known to experts. Recall that, for a system of admissible weights $\mathbf{w} = (w_1, \ldots, w_n)$, we can construct a measured neighborhood $N_{\mathbf{w}}(B)$ with a measured singular foliation $(\mathcal{F}, \mu)$. Let $\mathcal{F}_S$ be the union of the singular leaves of $\mathcal{F}$, and C_S the union of branch lines in $\mathcal{F}$ (Note that $\mathcal{F}$ does not have singularity of type triple points.) Then we construct abstract I-bundles $M(\mathcal{F}_S - C_S)$ ($M(C_S)$ resp.) with base space $\mathcal{F}_S - C_S$ (C_S resp.), where I-bundle structure coincides with the normal bundle structure on $\mathcal{F}_S$ in M (C_S in $\mathcal{F}_S$ resp.). Note that $M(\mathcal{F}_S - C_S)$ and $M(C_S)$ have not been embedded in M yet. Then let $M(\mathcal{F}_S)$ be the 3-manifold obtained from $M(C_S)$ by attaching $M(\mathcal{F}_S - C_S)$ so that $M(\mathcal{F}_S)$ looks like a total space of a measured neighborhood of $\mathcal{F}_S$ (see Figure 5.4). We use the following notations.

Let

$p : N_{\mathbf{w}}(B) \to B$ be the map giving the I-bundle structure on $N_{\mathbf{w}}(B)$,

$p_S : \mathcal{F}_S \to B$ the restriction of p to $\mathcal{F}_S$,

$p_{\mathcal{F}} : M(\mathcal{F}_S) \to \mathcal{F}_S$ the map giving the I-bundle structure inherited from those on $M(C_S)$ and $M(\mathcal{F}_S - C_S)$, where we suppose that $M(\mathcal{F}_S)$ is equipped with a Riemannian metric such that the length of the I-fibers become very swiftly short towards ends of $\mathcal{F}_S$.

Then let $\partial_h M(\mathcal{F}_S)$ be the subsurface of $\partial M(\mathcal{F}_S)$ corresponding to the ∂I-bundle.

Let $E(\mathcal{F}_S)$ be the metric completion of $M - \mathcal{F}_S$, and $\partial_h E(\mathcal{F}_S)$ the subsurface of $\partial E(\mathcal{F}_S)$ consisting of the completed points. Then there is a homeomorphism

$$f : \partial_h M(\mathcal{F}_S) \to \partial_h E(\mathcal{F}_S)$$

coming from the bundle structures, i.e., if $x \in \mathcal{F}_S$, then the boundary of the fiber $p_{\mathcal{F}}^{-1}(x)$ is mapped to the points of $\partial_h E(\mathcal{F}_S)$ corresponding to x in M respecting the normal bundle structure. Finally let M^* be the manifold obtained from $M(\mathcal{F}_S)$ and $E(\mathcal{F}_S)$ by identifying $\partial_h M(\mathcal{F}_S)$ and $\partial_h E(\mathcal{F}_S)$ by f, and $N^*(B)$ the image of $N_{\mathbf{w}}(B) \cup M(\mathcal{F}_S)$ in M^*.

For a point $x \in B$, let I_x^* be the image of $p^{-1}(x) \cup (p_S \circ p_{\mathcal{F}})^{-1}(x)$ in $N^*(B)$. Then

Claim. I_x^* is homeomorphic to the unit interval I.

Proof. Note that the number of the components of $\mathcal{F}_S$ is less than or equal to the number of the branch loci in B, hence it is finite. Note also that since $\mathcal{F}_S$ is "carried by B ", each component of $\mathcal{F}_S$ can be regarded as a union of countable number of copies of the sectors of B. These show that for each $x \in B$, $p^{-1}(x) \cap \mathcal{F}_S$ consists of countable number of points. Hence I_x^* is obtained from the I-fiber $p^{-1}(x)$ by inserting the components of $(p_S \circ p_{\mathcal{F}})^{-1}(x)$ at $p^{-1}(x) \cap \mathcal{F}_S$. Since the length of the I-fibers become very swiftly short towards ends of $\mathcal{F}_S$, we see that each I_x^* is homeomorphic to I

For example, we can define thickness of $M(\mathcal{F}_S)$ as below. We call the path-metric closure of each component of $\mathcal{F}_S - M(C_S)$ a *piece*. For every component of $\mathcal{F}_S$ we choose a single piece, and let $\mathcal{P}_0$ denote the set of such pieces. We inductively define a set of pieces $\mathcal{P}_i$ as below. We define $\mathcal{P}_i$ is the set of pieces which are adjacent to a piece of $\mathcal{P}_j$ and are not contained in $\mathcal{P}_j$ for $j < i$. Let $P_i = \cup \mathcal{P}_i$ be the union of the pieces of $\mathcal{P}_i$. Then $\mathcal{F}_S = \cup_n^\infty P_n$.

There is a positive integer A such that for every piece P the number of the pieces adjacent to P is less than A. We can choose A according to the branched surface B independently from $\mathcal{F}_S$. Note that the number of pieces of $\mathcal{P}_n$ is A^n. We can define thickness of $M(\mathcal{F}_S)$ so that the length of the I-fiber over each point of P_n is shorter than $1/(2A)^n$, and that that of $P_n \cap P_{n+1}$ is shorter than $1/(2A)^{n+1}$.

Let T be a positive integer such that the number of intersection points of any I-fiber and any piece is less than T. We can choose T according to B independently from $\mathcal{F}_S$. Then for each I-fiber I_x, the number of the intersection points $I_x \cap P_n$ is smaller than TA^n. Hence (the length of $M(\mathcal{F}_S) \cap I_x) \leq \sum_{n=0}^{\infty} \dfrac{1}{(2A)^n} \cdot TA^n = 2T < \infty$.

Moreover it is easy to see that these I_x^* give an I-bundle structure on $N^*(B)$, which is fiber preserving homeomorphic to $N(B)$ rel $\partial_h N(B)$. This shows that M^* is homeomorphic to M. Now we consider the image of $N_{\mathbf{w}}(B)$ in M^*. Here we note that some components of the image of $N_{\mathbf{w}}(B)$ are not smooth in a neighborhood of branch loci of B. Since such components are isolated from both sides, we can remove them to obtain a lamination, say $\mathcal{L}_{\mathbf{w}}$ in M^*. Then there is a transverse invariant measure $\mu_{\mathcal{L}}$ on $\mathcal{L}_{\mathbf{w}}$ induced from the transverse invariant measure on $N_{\mathbf{w}}(B)$, which represents the system of admissible weights $\mathbf{w}$. Note that $\mu_{\mathcal{L}}$ is 0 on $M(\mathcal{F}_S)$.

Conversely suppose a lamination $\mathcal{L}$ admits a transverse invariant measure μ. If $\mathcal{L}$ is carried by a branched surface B, then we can obtain a system of admissible weights on B from μ as follows.

> Let S be a sector of B, and J an I-fiber of the I-bundle $N(B)$ such that $J \cap (\text{Int } S) \neq \emptyset$. Then we define the weight on S by $\mu(J)$.

Note that the lengths of every pair of I-fibers in (a sector) $\times I$ are equal since the measure is invariant under translations along the leaves. It is easy to see that this defines a system of admissible weights on B.

In [O], it is shown that we can obtain all possible affine laminations carried by B via what are called broken invariant measures on B. Here we quickly see the method.

Let $\{S_1, \ldots, S_k\}$ be any set of transversely oriented properly embedded surfaces in M, which represents a basis for $H^1(M; \mathbb{R})$. We may suppose that $S_1, \ldots, S_k$, and B are in general position, and hence we obtain a branched surface B' with boundary by cutting B along the union of the surfaces $\cup_i S_i$. Then we consider a pair of arrays of positive real numbers $((\sigma_1, \ldots, \sigma_k), \mathbf{w})$, where $\mathbf{w}$ is a system of admissible weights on B'. We say that $((\sigma_1, \ldots, \sigma_k), \mathbf{w})$ is a *broken invariant measure* on B (for $S_1, \ldots, S_k$) if it satisfies the following condition.

Let Q_-, Q_+ be sectors of B' such that $Q_- \cap Q_+$ contains a 1-manifold, say ℓ, where $\ell \subset S_i$. Let $N(\ell)$ be a small regular neighbourhood of ℓ in $Q_- \cup Q_+$, and $N_+(\ell)$ (resp. $N_-(\ell)$) intersection of $N(\ell)$ and the $+$-side (resp. the $-$-side) of the surface S_i. Suppose that $N_+(\ell) \subset Q_+$ and $N_-(\ell) \subset Q_-$. Then we have $w_+ = \sigma_i w_-$, where $w_\pm$ denotes the weight on $Q_\pm$ in $\mathbf{w}$.

Then it is known that:

Proposition 1.3 of [O]**.** *Every broken invariant measure on B represents an affine lamination. Conversely, if $\mathcal{L}$ is a lamination carried by B which is affine in M, and a set of surfaces $S_1, \dots, S_k$ represents a basis for $H^1(M;\mathbb{R})$, then there is a broken invariant measure of B for $S_1, \dots, S_k$ which represents the affine lamination $\mathcal{L}$.*

Remark. We note that the correspondence between the affine structure and the broken invariant measure in Proposition 1.3. of [O] is natural. In fact, suppose there is a broken invariant measure on B for $S_1, \dots, S_k$. Let $p : \tilde{M} \to M$ be the universal cover. Let R_0 be an (arbitrarily fixed) component of $p^{-1}(M - \cup S_i)$. Let $\tilde{S}_i$ be the preimage of S_i in $\tilde{M}$. Then we define a transverse invariant measure on each component of $p^{-1}(B - \cup S_i)$ as follows.

Let R' be a component of $p^{-1}(M - \cup S_i)$ and B' the lift of a component B_0 of $B - \cup S_i$ contained in R'. We note that R' corresponds to an element $x_1[S_1] + \cdots + x_k[S_k] \in H^1(M;\mathbb{R})$ ($x_i \in \mathbb{Z}$), with R_0 regarded as representing the trivial element of $H^1(M;\mathbb{R})$. That is, if α is a path in $\tilde{M}$ from a point in $\mathrm{Int} R_0$ to $\mathrm{Int} R'$, then the algebraic intersection number of $p(\alpha)$ and S_i is x_i. Then we define the system of admissible weights on B' by $\sigma_1^{x_1} \cdots \sigma_k^{x_k} \mathbf{w}|_{B_0}$, where $\mathbf{w}|_{B_0}$ is the restriction of $\mathbf{w}$ on the sectors of B_0.

We can show that (see the proof of Proposition 1.3 of [O]) this system of admissible weights gives a system of admissible weights on $p^{-1}(B)$ which gives an affine structure on a lamination $\mathcal{L}$ carried by B.

Let $\{D_1^+, \dots, D_m^+\}$, $\{D_1^-, \dots, D_n^-\}$ be systems of generating disks for B_+, B_- respectively. We say that a positive system of admissible weights $\mathbf{w}^+ = (w_1^+, \dots, w_s^+)$ ($\mathbf{w}^- = (w_1^-, \dots, w_t^-)$ resp.) on τ_+ (τ_- resp.) is *positively induced from the system of generating disks* if there exists a system of positive real numbers $\{\alpha_1^+, \dots, \alpha_m^+\}$, ($\{\alpha_1^-, \dots, \alpha_n^-\}$ resp.) such that

$$\mathbf{w}^+ = \sum_{i=1}^{m} \alpha_i^+ \mathbf{b}_i^+ \quad (\mathbf{w}^- = \sum_{j=1}^{n} \alpha_j^- \mathbf{b}_j^- \text{ resp.}),$$

where $\mathbf{b}_i^+$ ($\mathbf{b}_j^-$ resp.) is the system of admissible weights on τ_+ (τ_- resp.) representing the simple closed curve ∂D_i^+ (∂D_j^- resp.).

Let $\tau_0 = \tau_+ \cap \tau_-$. We say that a pair of systems of admissible weights $\mathbf{w}^+ = (w_1^+, \ldots, w_s^+)$, $\mathbf{w}^- = (w_1^-, \ldots, w_t^-)$ on the train tracks τ_+, τ_- are *projectively attachable along* τ_0 if the following is satisfied.

> For each component f of τ_0, the systems of weights on f induced from $\mathbf{w}^+$, $\mathbf{w}^-$ are projectively equivalent, i.e., let r be the number of the edges of f, and $e_{i_1}^+, \ldots, e_{i_r}^+$, $e_{j_1}^-, \ldots, e_{j_r}^-$ the edges of τ_+, τ_- which are the edges of f with the same ordering. Then there exists a positive number c_f such that $(w_{i_1}^+, \ldots, w_{i_r}^+) = c_f \cdot (w_{j_1}^-, \ldots, w_{j_r}^-)$, where $w_i^\pm$ denotes the weight on the edge $e_i^\pm$ in $\mathbf{w}^\pm$.

Then we have,

Theorem 5.3. *Let B be a branched surface in standard position with respect to a diagram E of a link L, and let $\tau_\pm$, τ_0 be as above. Suppose each component of $B \cap B_+$, $B \cap B_-$ is simply connected. Then B fully carries a lamination $\mathcal{L}$ such that $\mathcal{L}$ is affine in $N(B)$ if and only if there exist a pair of positive systems of admissible weights $\mathbf{w}^+ = (w_1^+, \ldots, w_s^+)$, $\mathbf{w}^- = (w_1^-, \ldots, w_t^-)$ on τ_+, τ_- respectively which satisfy the following two conditions.*

1. *There exists a system of generating disks $\{D_1^+, \ldots, D_m^+\}$ ($\{D_1^-, \ldots, D_n^-\}$ resp.) for B_+ (B_- resp.) such that $\mathbf{w}^+ = (w_1^+, \ldots, w_s^+)$ ($\mathbf{w}^- = (w_1^-, \ldots, w_t^-)$ resp.) is positively induced from the system of generating disks.*
2. *The pair of the systems of admissible weights $\mathbf{w}^+$, $\mathbf{w}^-$ are projectively attachable along τ_0.*

Proof of only if part of Theorem 5.3. Let $p : \tilde{N}(B) \to N(B)$ be the universal cover. Suppose B fully carries a lamination $\mathcal{L}$ which is affine in $N(B)$.

> That is, there exists a transverse invariant measure μ on $\tilde{N}(B)$, and a homomorphism $\phi : \pi_1(N(B)) \to \mathbb{R}_+$ such that for each $\alpha \in \pi_1(N(B))$ we have:
>
> $$\alpha^*(\mu) = \phi(\alpha) \cdot \mu,$$
>
> where $\alpha^*(\mu)$ is the pull back measure of μ with α regarded as the covering translation corresponding to α.

Recall that we can obtain a positive system of admissible weights on the branched surface $p^{-1}(B)$ from μ.

Let $N_\pm = N(B) \cap B_\pm$. Here we may suppose that $N_\pm$ a union of I-fibers of $N(B)$. Note that $B_\pm$ is disjoint from the interior of the crossing balls. Since each component of $B \cap B_\pm$

is simply connected, each component of $p^{-1}(B \cap B_{\pm})$ is homeomorphic to a component of $B \cap B_{\pm}$. Hence there exists a lift $B \cap B_{\pm} \to \tilde{N}(B)$, which gives a homeomorphism onto the image, and we take an arbitrary one and fix it. By restricting the system of admissible weights μ on the image of $B \cap B_{\pm}$ by the lift, we obtain a positive system of admissible weights on $B \cap B_{\pm}$. Note that this system of weights varies according to the choice of the lift. Let $\mathcal{L}_{\pm} = \mathcal{L} \cap B_{\pm}$, and $\mu_{\pm}$ the transverse invariant measure on $\mathcal{L}_{\pm}$ induced by μ and representing the systems of weights. Let $\mathbf{w}^{\pm}$ be the positive system of admissible weights on $\tau^{\pm}$ induced from the system of admissible weights on $B \cap B_{\pm}$. By Proposition 5.1, there is a system of generating disks $\{D_1^+, \ldots, D_m^+\}$ ($\{D_1^-, \ldots, D_n^-\}$ resp.) for $B \cap B_+$ ($B \cap B_-$ resp.) which satisfies the following.

> Each leaf of $\mathcal{L}_+$ ($\mathcal{L}_-$ resp.) is isotopic to some D_i^+ (D_j^- resp.) in the I-bundle $N(B \cap B_+)$ ($N(B \cap B_-)$ resp.) by a fiber preserving isotopy.

Let $\mathcal{D}_i^+$ ($\mathcal{D}_j^-$ resp.) be the union of the leaves of $\mathcal{L} \cap B_+$ ($\mathcal{L} \cap B_-$ resp.) which are isotopic to D_i^+ (D_j^- resp.) by fiber preserving isotopies in the I-bundle $N(B \cap B_+)$ ($N(B \cap B_-)$ resp.). Recall that $\mathcal{D}_i^+$ ($\mathcal{D}_j^-$ resp.) is a closed subset of B_+ (B_- resp.) by Proposition 5.1. Let

$$\alpha_i^+ = \max\{\mu_+(J) | J \text{ is a subinterval of a fiber of } N(B) \text{ such that } \partial J \subset \mathcal{D}_i^+\},$$

$$\alpha_j^- = \max\{\mu_-(J) | J \text{ is a subinterval of a fiber of } N(B) \text{ such that } \partial J \subset \mathcal{D}_j^-\}.$$

That is, α_i^+ (α_j^- resp.) is the "thickness "of $\mathcal{D}_i^+$ ($\mathcal{D}_j^-$ resp.). Since $\mathcal{L} \cap B_{\pm}$ is a support of the measure $\mu_{\pm}$, we see that $\alpha_i^+ > 0$ ($\alpha_j^- > 0$ resp.). Since the measure is invariant under translations along the leaves, we have

$$\mathbf{w}^+ = \sum_{i=1}^{m} \alpha_i^+ \mathbf{b}_i^+, \mathbf{w}^- = \sum_{j=1}^{m} \alpha_j^- \mathbf{b}_j^-,$$

where $\mathbf{b}_i^+$, $\mathbf{b}_j^-$ are the systems of weights representing simple closed curves ∂D_i^+, ∂D_j^- carried by τ_+, τ_- respectively. This shows that $\mathbf{w}^+$ ($\mathbf{w}^-$ resp.) is positively induced from the system of generating disks $\{D_1^+, \ldots, D_m^+\}$ ($\{D_1^-, \ldots, D_n^-\}$ resp.).

Let f be a component of τ_0. We note that the weight on f in $\mathbf{w}^{\pm}$ is that of a component of $p^{-1}(f)$ determined by the measure μ. On the other hand, the weights on the components of $p^{-1}(f)$ are mutually projectively equivalent since $\alpha^*(\mu) = \phi(\alpha) \cdot \mu$, for each $\alpha \in \pi_1(M)$. Hence we see that the systems of weights on f induced from $\mathbf{w}^+$ and $\mathbf{w}^-$ are projectively equivalent. Thus the systems of admissible weights $\mathbf{w}^+$, $\mathbf{w}^-$ are projectively attachable along τ_0. This completes the proof of only if part of Theorem 5.3.

Proof of if part of Theorem 5.3. Suppose there exist systems of admissible weights $\mathbf{w}^+ = (w_1^+, \ldots, w_m^+)$ and $\mathbf{w}^- = (w_1^-, \ldots, w_n^-)$ on the train tracks τ_+ and τ_- respectively which are projectively attachable along τ_0. Recall that we have $\mathbf{w}^+ = \sum_{i=1}^m \alpha_i^+ \mathbf{b}_i^+, \mathbf{w}^- = \sum_{j=1}^n \alpha_j^- \mathbf{b}_j^-$, where $\mathbf{b}_i^+$, $\mathbf{b}_j^-$ are the systems of weights representing simple closed curves ∂D_i^+, ∂D_j^- carried by τ_+, τ_- respectively. Here $B \cap B_+$ ($B \cap B_-$ resp.) is a pinching of $D_1^+ \cup \cdots \cup D_m^+$ ($D_1^- \cup \cdots \cup D_n^-$ resp.). We may regard the weight α_i^+ (α_j^- resp.) is assigned to D_i^+ (D_j^- resp.). On each sector of $B \cap B_+$ ($B \cap B_-$ resp.) the weights on the generating disks intersecting the sector sum up to the weight on the sector. Then we obtain the system of weights on $B \cap B_\pm$. Let $N_\pm$ be the foliated regular neighborhood of $B \cap B_\pm$ with transverse invariant measure corresponding to $\mathbf{w}^\pm$, and $\mathcal{F}_\pm$ the corresponding singular foliation on $N_\pm$.

Since the systems of weights $\mathbf{w}^+$ and $\mathbf{w}^-$ on the train tracks τ_+ and τ_- are projectively attachable along τ_0, we may suppose that $\mathcal{F}_+ \cap S_0 = \mathcal{F}_- \cap S_0$, where the transverse invariant measures on $\mathcal{F}_+$ and $\mathcal{F}_-$ are matched linearly in each component of $N_\pm \cap S_0$. Let $\mathcal{F}^*$ be the singular foliation $\mathcal{F}_+ \cup \mathcal{F}_-$ on $N^* = N_+ \cup N_-$.

Let $f_1, \ldots, f_l$ be the components of $N^* \cap S_0$. Since each component of $B \cap B_\pm$ is simply connected, using Van Kampen's theorem, we may suppose (by changing suffix if necessary) that there exists an integer $k (< l)$ such that

(1) the manifold obtained from the disjoint union of N_+ and N_- by pasting them along $\cup_{i=k+1}^l f_i$ is connected and simply connected and
(2) for each j $(1 \leq j \leq k)$, the manifold obtained from the disjoint union of N_+ and N_- by pasting them along $f_j \cup (\cup_{i=k+1}^l f_i)$ is not simply connected.

Note that the system of surfaces $f_1, \ldots, f_k$ represents a generator system of $H^1(N^*; \mathbb{R})$. Then we have:

Claim 1. $\mathcal{F}^*$ has an affine structure as a singular foliation in N^*.

Proof. Let N_0^* be the manifold obtained from the disjoint union of N_+ and N_- by pasting them along $\cup_{i=k+1}^l f_i$. Since N_0^* is simply connected, by multiplying the transverse invariant measures on components of N_+, N_- by positive constant numbers if necessary, we may suppose that the measures coincide on $f_{k+1}, \ldots, f_l$. Hence we obtain a transverse invariant measure on N_0^*.

Then we can obtain a broken invariant measure on N^* by using the surfaces $f_1, \ldots, f_k$ and the above measure on N_0^*. By the above-mentioned Proposition 1.3 of [O], we see that $\mathcal{F}^*$ has an affine structure in N^*.

Let D^3 be a crossing ball. By the definitions of $\mathcal{F}^*$ and N^*, we see that each component of $\partial D^3 \cap N^*$ is an annulus, which is a union of four trapezoids such that two of them are on S_+ and the other two are on S_-.

$\boxed{Figure\ 5.5}$

Claim 2. For each component A of $\partial D^3 \cap N^*$, $\mathcal{F}^* \cap A$ is a product foliation with each leaf a circle.

Proof. Let e_p^+, e_q^+ (e_r^-, e_s^- resp.) be the edges of τ_+ (τ_- resp.) intersecting A. Let w_p^+, w_q^+ (w_r^-, w_s^- resp.) be the weights on e_p^+, e_q^+ (e_r^-, e_s^- resp.) in $\mathbf{w}_+$ ($\mathbf{w}_-$ resp.). We start at a point in $A \cap e_p^+$ and go around A to come back to the starting point. Then the width of A is changed as $w_p^+ \to w_r^- \to w_q^+ \to w_s^-$. Hence the holonomy of $\mathcal{F}^* \cap A$ along ∂A is represented by the affine map

$$x \to \left(\frac{w_r^-}{w_p^+} \cdot \frac{w_q^+}{w_r^-} \cdot \frac{w_s^-}{w_q^+} \cdot \frac{w_p^+}{w_s^-} \right) x = x.$$

This shows that $\mathcal{F}^* \cap A$ is a product foliation, with each leaf parallel to a component of ∂A.

By Claim 2, we can insert (saddles)$\times I$ in the crossing balls to cap off the foliated annuli, and we obtain a singular foliation $\mathcal{F}$ without boundary in $N(B)$.

Claim 3. $\mathcal{F}$ has an affine structure as a singular foliation in $N(B)$.

Proof. By Claim 1, $\mathcal{F} \cap N^*$ has an affine structure as a singular foliation in N^*, i.e.,

> [1] Let $p_0 : \tilde{N}^* \to N^*$ be the universal cover. Then there is a transverse invariant measure μ_0 on the singular foliation $p_0^{-1}(\mathcal{F} \cap N^*)$ and a homomorphism
>
> $$\phi_0 : \pi_1(N^*) \to \mathbb{R}_+$$
>
> such that, for each $\alpha \in \pi_1(N^*)$, we have
>
> $$\alpha^*(\mu_0) = \phi_0(\alpha) \cdot \mu_0,$$
>
> where $\alpha^*(\mu_0)$ is the pull back measure of μ_0 with α regarded as a covering translation.

Note that the transverse invariant measure μ_0 is not broken. Let H be the normal subgroup of $\pi_1(N^*)$ generated by the fundamental groups of the components of $N^* \cap$ (the bubbles) the union of the annuli. By Van-Kampen's theorem, we see that $\pi_1(N(B)) \cong \pi_1(N^*)/H$. Hence we have $\tilde{N}(B) - p^{-1}$(the crossing balls) $= \tilde{N^*}/H$, where $p : \tilde{N}(B) \to N(B)$ is the universal cover. By the proof of Claim 2, we see that for each $h \in H$, we have $\phi_0(h) = 1$, and this shows that (1) μ_0 projects to a transverse invariant measure, say μ', on $\tilde{N}(B) - p^{-1}$(the interior of the crossing balls), and (2) ϕ_0 projects to a homomorphism $\phi' : \pi_1(N^*)/H \cong \pi_1(N(B)) \to \mathbb{R}_+$. Since $\mathcal{F} \cap$ (the crossing balls) is a product foliation, and since transverse invariant measures are invariant under translations along leaves, the measure μ' on $p^{-1}(\mathcal{F} -$ (the interior of the crossing balls)) is uniquely extended to a transverse invariant measure on $p^{-1}(\mathcal{F})$, say $\mu_{\mathcal{F}}$. Then, by the above [1] and the properties of ϕ_0 above, we see that $\mu_{\mathcal{F}}$ together with ϕ' gives a transverse affine structure on $\mathcal{F}$ in $N(B)$.

Let $\mathcal{L}$ be a lamination obtained by splitting $\mathcal{F}$ along the singular leaves. By Claim 3, we see that $\mathcal{L}$ has an affine structure as a lamination in $N(B)$, and this completes the proof of if part of Theorem 5.3.

For the statement of Theorem 5.4, we prepare some terminologies.

In general, let τ be a train track embedded in a surface F, and τ' a subset of τ such that each component of τ' is a train track, and that each component of $\mathrm{cl}(\tau - \tau')$ is an arc contained in the interior of an edge of τ. We call τ' a *broken train track (obtained from τ)*.

We may suppose that each fiber of $N(\tau')$ is a fiber of $N(\tau)$. Let $\mathbf{w}$ be a system of admissible weights on τ', γ a simple closed curve in $N(\tau)$ which intersects each fiber of the I-bundle $N(\tau)$ at no more than one point, i.e., γ is isotoped to be embedded in τ.

Remark. Note that since $\mathbf{w}$ is a system of admissible weights on τ', the weight w_1 on an edge e_1 and the weight w_2 on an edge e_2 may differ even if e_1 and e_2 are contained in the same edge of τ. Note also that what is required on the weights $\mathbf{w}$ on τ' is just the switch condition at each vertex of τ' which is a vertex of τ.

We say that $\mathbf{w}$ is *compatible* with γ if it satisfies the following condition.

> Take a base point in the interior of an edge of τ' contained in γ and track γ around. Let $a_1, \ldots, a_n$ be the components of $\mathrm{cl}(\gamma - \tau')$ which we pass in this order, and let $\partial_- a_i$ ($\partial_+ a_i$ resp.) denote the end point of a_i through which we enter (leave resp.) a_i. Let $w_i^\pm$ be the weight on the edge of τ' containing $\partial_\pm a_i$. Then we have;
>
> $$\left(\frac{w_1^+}{w_1^-}\right)\left(\frac{w_2^+}{w_2^-}\right)\cdots\left(\frac{w_n^+}{w_n^-}\right) = 1.$$

Remark. This definition does not depend on the choice of the base point.

Suppose $\mathbf{w}$ is compatible with γ. Let $e_{i_1}, \ldots, e_{i_m}$ be the edges of τ' through which γ goes successively, with starting point $p \in e_{i_1}$.

We define a system of admissible weights $\mathbf{a}$ on τ' inductively as follows.

> We set the weight on e_{i_1} in $\mathbf{a}$, say a_{i_1}, to be an arbitrarily fixed positive real number. Suppose we have defined the weight on e_{i_k} in $\mathbf{a}$, say a_{i_k}. Then we define the weight on $e_{i_{k+1}}$ in $\mathbf{a}$, say $a_{i_{k+1}}$, as below.
>
> (1) If $e_{i_k} \cap e_{i_{k+1}} \neq \emptyset$ (, i.e., $e_{i_k} \cap e_{i_{k+1}}$ is a switch of τ), then
>
> $$a_{i_{k+1}} = a_{i_k}.$$
>
> (2) If $e_{i_k} \cap e_{i_{k+1}} = \emptyset$, then
>
> $$a_{i_{k+1}} = \left(\frac{w_{i_{k+1}}}{w_{i_k}}\right) a_{i_k},$$
>
> where w_j denotes the weight on the edge e_j in $\mathbf{w}$.
>
> Finally, we set the weight on e_j ($j \neq i_1, \ldots, i_m$) in $\mathbf{a}$ to be equal to 0.

We say that $\mathbf{a}$ is (a system of weights) *induced from* $\mathbf{w}$ *to represent* (the simple closed curve) γ.

Remark. The system of weights $\mathbf{a}$ is not uniquely determined by γ. In fact, it depends on the choice of the starting point p, and the weight on the edge e_{i_1} containing the starting point. However, since $\mathbf{w}$ is compatible with γ, it is easy to see that the systems of weights are mutually projectively equivalent, i.e., if $\mathbf{a}$ and $\mathbf{a}'$ are induced from $\mathbf{w}$ to represent γ, then there is a constant real number $c > 0$ such that $\mathbf{a} = c\mathbf{a}'$.

Let $\gamma_1, \dots, \gamma_p$ be mutually disjoint simple closed curves in F such that $\gamma_1 \cup \cdots \cup \gamma_p$ is carried by τ. We say that $\mathbf{w}$ is *positively generated by* $\gamma_1, \dots, \gamma_p$, if

(1) for each i ($i = 1, \dots, p$), $\mathbf{w}$ is compatible with γ_i, and if
(2) there are systems of weights $\mathbf{a}_1, \dots, \mathbf{a}_p$ induced from $\mathbf{w}$ to represent $\gamma_1, \dots, \gamma_p$ such that $\mathbf{w} = \Sigma_{i=1}^p \mathbf{a}_i$.

We return to our situation. That is, B is a branched surface in standard position with respect to a diagram E of a link L, and $\tau_\pm$ the train track $B \cap S_\pm$. Let $\tau_0 = \tau_+ \cap S_0 (= \tau_- \cap S_0)$. Then we define a subset τ' of τ_0 as follows.

> In the interior of each edge which is not incident to a bubble, take a point, called a *break point*. Then remove from τ_0;
>
> (1) sufficiently small neighborhoods of the break points and
> (2) every edge of τ_0 which has both of its end points on the bubbles.

Note that τ' can be regarded as a broken train track obtained from τ_+ (τ_- resp.). Then we have:

Theorem 5.4. *Let B be a branched surface in standard position with respect to a diagram E of a link L, and let $\tau_\pm$, τ_0, τ' be as above. Suppose B fully carries a lamination $\mathcal{L}$ such that $\mathcal{L}$ is affine in $N(B)$, then*

(*) there exists a system of admissible weights $\mathbf{w}'$ on the broken train track τ' such that there exist systems of generating disks $\{D_1^+, \dots, D_m^+\}$ and $\{D_1^-, \dots, D_n^-\}$ for $B \cap B_+$ and $B \cap B_-$ respectively which satisfies the following.

(1) *$\mathbf{w}'$ is positively generated by $\partial D_1^+, \dots, \partial D_m^+$ with τ' regarded as a broken train track obtained from τ_+ and*
(2) *$\mathbf{w}'$ is positively generated by $\partial D_1^-, \dots, \partial D_n^-$ with τ' regarded as a broken train track obtained from τ_-.*

Conversely, if (*) holds, then there exist a branched surface $\hat{B}$ and a lamination $\hat{\mathcal{L}}$ fully carried by $\hat{B}$ such that $\hat{B}$ is obtained by splitting B in Int $(B_+) \cup$ Int (B_-) and such that $\hat{\mathcal{L}}$ is an affine lamination in $N(\hat{B})$ with the affine structure given by $\mathbf{w}'$.

Remark. In the latter half of Theorem 5.4, we note that $\hat{B} \cap S_\pm = \tau_\pm = B \cap S_\pm$, since the splitting is performed in Int $(B_+) \cup$ Int (B_-), and the lamination $\hat{\mathcal{L}}$ is fully carried also by B.

Proof of the former half of Theorem 5.4. Let $S_1, \dots, S_k$ be surfaces each of which is properly embedded in $N(B)$ so that they are in general position and together represent a basis for $H^1(N(B); \mathbb{R})$. Since B is a deformation retract of $N(B)$, we may suppose that $S_1, \ldots, S_k$ are unions of I-fibers of the I-bundle $N(B)$. Let $K_1, \ldots, K_k$ be the 1-complexes in B which are the images of $S_1, \ldots, S_k$ by the projection map $N(B) \to B$. We may suppose that the 1-complexes $K_1, \ldots, K_k$ are in general position in B and that $\cup K_i$ intersects the projection 2-sphere S in finitely many transverse points away from the bubbles.

Suppose B fully carries a lamination $\mathcal{L}$ which is affine in $N(B)$. By Proposition 1.3 of [O], there exists a broken invariant measure $((\sigma_1, \ldots, \sigma_k), \mathbf{w}_b)$ on B for $S_1, \ldots, S_k$ giving the affine structure. Recall that, by the note immediately after the definition of transverse affine structure, we can obtain a measured lamination from a system of admissible weights on a branched surface.

Let τ_* be the broken train track obtained from τ_0 by removing a sufficiently small regular neighborhood of $(\cup K_i) \cap S_0$. Let $\mathbf{w}_*$ denote the system of admissible weights on τ_* induced from $\mathbf{w}_b$.

Let $\mathcal{L}_\pm = \mathcal{L} \cap B_\pm$. By Proposition 5.1, there is a system of generating disks $\{D_1^+, \ldots, D_m^+\}$ ($\{D_1^-, \ldots, D_n^-\}$ resp.) for $B \cap B_+$ ($B \cap B_-$ resp.) as below.

> Each leaf of $\mathcal{L}_+$ ($\mathcal{L}_-$ resp.) is isotopic to some D_i^+ (D_j^- resp.) in the I-bundle $N(B \cap B_+)$ ($N(B \cap B_-)$ resp.) by a fiber preserving isotopy.

Let $\mathcal{D}_i^+$ ($\mathcal{D}_j^-$ resp.) be the union of the leaves of $\mathcal{L}_+$ ($\mathcal{L}_-$ resp.) which are isotopic to D_i^+ (D_j^- resp.) by a fiber preserving ambient isotopy in $N(B \cap B_+)$ ($N(B \cap B_-)$ resp.). Let Q be a component of $D_i^+ - (\cup S_i)$ ($D_j^- - (\cup S_i)$ resp.), and $\mathcal{D}_i^+(Q)$ ($\mathcal{D}_j^-(Q)$ resp.) be the union of the components of $\mathcal{D}_i^+ - (\cup S_i)$ ($\mathcal{D}_j^- - (\cup S_i)$ resp.) which are isotopic to Q by a fiber preserving ambient isotopy in $N(B \cap B_+)$ ($N(B \cap B_-)$ resp.). Then by Proposition 5.1, we see that for each fiber J of $N(B \cap B_+)$ ($N(B \cap B_-)$ resp.) with $J \cap \mathcal{D}_i^+(Q) \neq \emptyset$ ($J \cap \mathcal{D}_j^-(Q) \neq \emptyset$ resp.) the intersection $J \cap \mathcal{D}_i^+(Q)$ ($J \cap \mathcal{D}_j^-(Q)$ resp.) is a closed subset of J.

According to this observation, let

$m_i^+(Q) =$

$\max\{\mu_b(J) | J$ is a subinterval of a fiber of $N(B)$ such that $\partial J \subset \mathcal{D}_i^+$ and $J \cap Q \neq \emptyset\}$,

$m_j^-(Q) =$

$\max\{\mu_b(J) | J$ is a subinterval of a fiber of $N(B)$ such that $\partial J \subset \mathcal{D}_j^-$ and $J \cap Q \neq \emptyset\}$,

where μ_b is the broken invariant measure determined by the system of weights $\mathbf{w}_b$. That is, $m_i^+(Q)$ ($m_j^-(Q)$ resp.) is the "thickness" of $\mathcal{D}_i^+$ ($\mathcal{D}_j^-$ resp.) in Q. Since $\mathcal{L} - (\cup S_i)$ is the support of the measure, we see that $m_i^+(Q) > 0$ ($m_j^-(Q) > 0$ resp.).

Note that $Q \cap \tau_*$ is a union of edges of τ_*. Then let $\mathbf{c}_i^+$ be a system of admissible weights on τ_* obtained from D_i^+ as follows.

(1) If e is an edge of τ_* contained in a component Q of $D_i^+ - (\cup S_l)$, then we assign $m_i^+(Q)$ to e.

(2) If e is an edge of τ_* not contained in a component of $D_i^+ - (\cup S_l)$, then we assign 0 to e.

We can analogously define $\mathbf{c}_j^-$.

Then we obviously have the equations below.

[2] $$\mathbf{w}_* = \sum \mathbf{c}_i^+ = \sum \mathbf{c}_j^-$$

By definition, it is easy to see that each ∂D_i^+ (∂D_j^- resp.) is compatible with $\mathbf{w}_*$, and that $\mathbf{c}_i^+$ ($\mathbf{c}_j^-$ resp.) is a system of weights induced from $\mathbf{w}$ to represent the simple closed curve ∂D_i^+ (∂D_j^- resp.).

Hence, by the above equation [2], we see that

> $\mathbf{w}_*$ is positively generated by $\partial D_1^+, \ldots, \partial D_m^+$ ($\partial D_1^-, \ldots, \partial D_n^-$ resp.) with τ_* regarded as a broken train track obtained from τ_+ (τ_- resp.).

In general, for a system of admissible weights $\mathbf{v}$ on τ_*, we define a system of admissible weights, denoted by $b(\mathbf{v})$, on τ' as follows.

(1) Let e be an edge of τ_0 which is not incident to a bubble. Then there is a break point in the interior of e, and e is separated into two edges, say e_1, e_2, in τ'.

In this case, let $f_1, \ldots, f_q$ ($q \geq 1$) be the closures of the components of $e-(\cup K_l)$ which are located in e in this order, where f_1 contains the endpoint $\partial e_1 \cap \partial e$ and f_q contains the endpoint $\partial e_2 \cap \partial e$. Then we set the weight on e_1 (e_2 resp.) in $b(\mathbf{v})$ to be equal to that on f_1 (f_q resp.) in $\mathbf{v}$.

(2) Let e be an edge of τ_0 such that one endpoint of e is contained in a bubble and the other endpoint is a switch. (Hence e is embedded in τ'.)

In this case, let $f_1, \ldots, f_q$ ($q \geq 1$) be the closures of the components of $e-(\cup K_i)$ which are located in e in this order so that f_1 is incident to the bubble and f_q is incident to the switch. Then we let the weight on e be equal to the weight on f_q in $\mathbf{v}$.

Then we let $\mathbf{w}' = b(\mathbf{w}_*)$, $\mathbf{a}_i^+ = b(\mathbf{c}_i^+)$ and $\mathbf{a}_j^- = b(\mathbf{c}_j^-)$.

Then we obviously have the equation below.

[3] $$\mathbf{w}' = \sum \mathbf{a}_i^+ = \sum \mathbf{a}_j^-$$

Claim 1. For each D_i^+, D_j^- above, $\mathbf{w}'$ is compatible with ∂D_i^+, ∂D_j^- and $\mathbf{a}_i^+$ ($\mathbf{a}_j^-$ resp.) is induced from $\mathbf{w}'$ to represent ∂D_i^+ (∂D_j^- resp.).

Proof. Since the argument is the same, we show this for ∂D_i^+.

Note that ∂D_i^+ is embedded in τ_+. Let g be the closure of a component of $\partial D_i^+ - \tau'$. Then let e_1, e_2 be the edges of τ' which are incident to g. Let $e = e_1 \cup g \cup e_2$. Then e is an edge of τ_+. Let $f_1, \dots, f_q$ ($q \geq 1$) be the closures of the components of $e - ((\cup K_l) \cup (\text{the interiors of the crossing balls})$ which are located in e in this order, where f_1 contains the endpoint $\partial e_1 \cap \partial e$ and f_q contains the endpoint $\partial e_2 \cap \partial e$. Let $v_1, \dots, v_q$ be the weights on $f_1, \dots, f_q$ respectively in $\mathbf{w}_*$. Then the ratio of the affine map induced by $\mathbf{w}_*$ when we track e from f_1 to f_q is $(v_2/v_1)(v_3/v_2)\cdots(v_q/v_{q-1}) = v_q/v_1$. On the other hand, the ratio of the affine map induced by $\mathbf{w}'$ when we track e from e_1 to e_2 is v_q/v_1, which is exactly the same as above. It is easy to see that this implies Claim 1 since ∂D_i^+ is compatible with $\mathbf{w}_*$.

Claim 1 together with above [3] implies the former half of Theorem 5.4

Proof of the latter half of Theorem 5.4. Suppose there exists a system of admissible weights $\mathbf{w}'$ on the broken train track τ', and systems of generating disks $\{D_1^+, \dots, D_m^+\}$ and $\{D_1^-, \dots, D_n^-\}$ for $B \cap B_+$ and $B \cap B_-$ respectively which satisfies condition (*) of Theorem 5.4.

Let τ'' be the train track contained in τ_0 such that $\tau'' = \tau' \cup$(the edges of τ_0 each of whose endpoints is contained in bubbles).

Then let $\mathbf{w}''$ be the system of weights on τ'' obtained as below.

(1) If e is an edge of τ'' which is an edge of τ', then we assign the weight on e in $\mathbf{w}'$ to e.
(2) If e is an edge of τ'' which is not an edge of τ', then we assign 1 to e.

It is easy to see that $\mathbf{w}''$ is positively generated by $\partial D_1^+, \dots, \partial D_m^+$ with τ'' regarded as a subset of τ_+, and that $\mathbf{w}''$ is positively generated by $\partial D_1^-, \dots, \partial D_n^-$ with τ'' regarded as a subset of τ_-.

We recall the construction of the foliated neighborhood $N_{\mathbf{w}''}(\tau'')$ with a transverse invariant measure corresponding to $\mathbf{w}''$. Let $e_1'', \cdots, e_h''$ be the edges of τ'', and $w_1'', \cdots, w_h''$ the weights on these edges in $\mathbf{w}''$. Then $N_{\mathbf{w}''}(\tau'')$ is the union of I-bundles $e_i'' \times [0, w_i'']$ foliated by the leaves of the form $e_i'' \times$(a point). Then $N(\tau'')$ is foliated by the leaves which are unions of the leaves of the above form. Note that this foliation has singular leaves which intersect singular points in $\partial N_{\mathbf{w}''}(\tau'')$. Let t be a vertex of τ'' of valency 1. Then the subarc of $\partial N(\tau'')$ corresponding to $t \times [0, w_i'']$ is called a *terminal boundary* of $N(\tau')$.

See Figure 5.6. Then we connect terminal boundaries of $N_{\mathbf{w}''}(\tau'')$ in neighborhoods of the break points and components of $\tau_\pm \cap$ (bubbles) by using fibered "trapezoids"as in Figure 5.6 so that the transverse invariant measures are matched by affine maps. Let N_1 be the resulting 2-complex with a singular foliation $\mathcal{F}_1$. (Note that N_1 is a "fibered neighborhood"of $\tau_+ \cup \tau_-$.)

Figure 5.6

Since $\mathcal{F}_1$ is obtained from the measured singular foliation on $N_{\mathbf{w}''}(\tau'')$ by pasting the measures by affine maps, we can show, by similar arguments as in the proof of Proposition 1.3 of [O], that $\mathcal{F}_1$ admits an affine structure, i.e.,

> [4] Let $p_1 : \tilde{N}_1 \to N_1$ be the universal cover. There exists a transverse invariant measure $\tilde{\mu}_1$ on $p_1^{-1}(\mathcal{F}_1)$, and a homomorphism $\tilde{\phi}_1 : \pi_1(N_1) \to \mathbb{R}_+$ such that for each $\alpha \in \pi_1(N_1)$ we have:
>
> $$\alpha^*(\tilde{\mu}_1) = \tilde{\phi}_1(\alpha) \cdot \tilde{\mu}_1.$$

We omit the proof.

Let D^3 be a crossing ball. By the definition of N_1, we see that each component of $\partial D^3 \cap N_1$ is an annulus, which is a union of four trapezoids such that two of them are on S_+ and the other two are on S_-.

Figure 5.7

Claim 1. For each component A of $\partial D^3 \cap N_1$, $\mathcal{F}_1 \cap A$ is a product foliation with each leaf a circle.

Proof. Let e''_p, e''_q, e''_r, e''_s be the edges of τ'' intersecting A at their endpoints. We start at the point in $A \cap e''_p$ and go around A to come back to the starting point. Then the width of A is changed as $w''_p \to w''_r \to w''_q \to w''_s$, where w''_i denotes the weight on e''_i in $\mathbf{w}''$. Hence the holonomy of $\mathcal{F}_1 \cap A$ along ∂A is represented by the affine map

$$x \to \left(\frac{w''_r}{w''_p} \cdot \frac{w''_q}{w''_r} \cdot \frac{w''_s}{w''_q} \cdot \frac{w''_p}{w''_s} \right) x = x.$$

This shows that $\mathcal{F}_1 \cap A$ is a product foliation, with each leaf parallel to a component of ∂A.

By Claim 1, we can insert (saddles) $\times I$ in the crossing balls to cap off the foliated annuli, and we obtain a 3-complex, say N_2, with a "singular foliation", say $\mathcal{F}_2$. Let $N_2^{\pm} = N_2 \cap S_{\pm} (= N_1 \cap S_{\pm})$, and $\mathcal{F}_2^{\pm}$ the foliation on $N_2^{\pm}$ obtained by restricting $\mathcal{F}_2$ on $N_2^{\pm}$.

Since $\mathbf{w}''$ is positively generated by $\{\partial D_1^+, \dots, \partial D_m^+\}$ ($\{\partial D_1^-, \dots, \partial D_n^-\}$ resp.), we see that each non-singular leaf of $\mathcal{F}_2^+$ ($\mathcal{F}_2^-$ resp.) is compact and parallel to some ∂D_i^+ (∂D_j^- resp.) in N_2^+ (N_2^- resp.). Let A_i^+ (A_j^- resp.) be the closure of the union of non-singular leaves of $\mathcal{F}_2^+$ ($\mathcal{F}_2^-$ resp.) that are parallel to ∂D_i^+ (∂D_j^- resp.) in N_2^+ (N_2^- resp.). Let N_2' be a 3-manifold obtained from a disjoint union of N_2, $N_2^+ \times I$ and $N_2^- \times I$ by identifying N_2^+, N_2^- with $N_2^+ \times \{0\}$, $N_2^- \times \{0\}$ respectively. Then $\mathcal{F}_2$ and the product foliations $\mathcal{F}_2^+ \times I$, $\mathcal{F}_2^- \times I$ are joined to give a foliation, say $\mathcal{F}_2'$, on N_2'. Let $A_i^{+\prime}$ ($A_j^{-\prime}$ resp.) be the annulus in $\partial N_2'$ corresponding to $A_i^+ \times \{1\}$ ($A_j^- \times \{1\}$ resp.). Let $\hat{N}$ be the 3-manifold obtained from a disjoint union of $D_1^+ \times I, \dots, D_m^+ \times I$, $D_1^- \times I, \dots, D_n^- \times I$ and N_2' by identifying $\partial D_1^+ \times I, \dots, \partial D_m^+ \times I$, $\partial D_1^- \times I, \dots, \partial D_n^- \times I$ and $A_1^{+\prime}, \dots, A_m^{+\prime}$, $A_1^{-\prime}, \dots, A_n^{-\prime}$ respectively. Here we may suppose that the foliation $\mathcal{F}_2'$ and the product foliations on $D_1^+ \times I, \dots, D_m^+ \times I$, $D_1^- \times I, \dots, D_n^- \times I$ are matched to give a foliation, say $\hat{\mathcal{F}}$, on $\hat{N}$.

Claim 2. $\hat{\mathcal{F}}$ admits an affine structure.

Proof. Let $\hat{p} : \tilde{\hat{N}} \to \hat{N}$ be the universal cover. Let $N_1' = \mathrm{cl}(N_2' - ($ the interior of the crossing balls$))$. Let H be the normal subgroup of $\pi_1(N_1')$ generated by the fundamental groups of $A_1^{+\prime}, \dots, A_m^{+\prime}$, $A_1^{-\prime}, \dots, A_n^{-\prime}$ and the fundamental groups of the annuli $N_1 \cap ($ the crossing balls$)$. By applying Van-Kampen's Theorem successively, we see that $\pi_1(\hat{N}) \cong \pi_1(N_1')/H$. By Claim 1 and the fact that the restrictions of $\mathcal{F}_2'$ on $A_1^{+\prime}, \dots, A_m^{+\prime}$, $A_1^{-\prime}, \dots, A_n^{-\prime}$ and the annuli $N_2' \cap$(the bubbles) are product foliations, $\tilde{\phi}_1(h) = 1$ for each $h \in H$. (For the definition of $\tilde{\phi}_1$ and $\tilde{\mu}_1$, see [4] above.) This shows that (1) $\tilde{\mu}_1$ projects to a transverse invariant measure, say μ', on $\hat{p}^{-1}(N_1')$, and that (2) $\tilde{\phi}_1$ projects to a homomorphism $\hat{\phi} : \pi_1(N_1')/H \cong \pi_1(\hat{N}) \to \mathbb{R}_+$. Since the restriction of $\hat{\mathcal{F}}$ on each component of $\hat{N} - N_1'$ is a product foliation of the form (open disk)$\times I$, the measure μ' is uniquely extended to a transverse invariant measure, say $\hat{\mu}$ on $\tilde{\hat{N}}$ so that $\hat{\mu}$ is invariant under translations along leaves. Then, by the above [4] and the properties of $\tilde{\phi}_1$ above, we see that $\hat{\mu}$ together with $\hat{\phi}$ gives a transverse affine structure on $\hat{\mathcal{F}}$. This completes the proof of Claim 2.

We note that $\hat{N}$ is embedded in the exterior $E(L)$ of the link, that is, $\hat{N} \cap (S_+ \cup S_-) = N_1$. We also note that $\hat{N}$ has an I-bundle structure which is an extension of an I-bundle structure on N_1. By collapsing each fiber of the I-bundle structure on $\hat{N}$ to a point, we obtain a branched surface with non-generic branch locus. Then we slightly perturb it to obtain a branched surface $\hat{B}$ in $E(L)$. Note that $\hat{B}$ is obtained from B by splitting in

Int $(B_+) \cup$ Int (B_-). Let $\mathcal{L}$ be a lamination obtained by splitting $\hat{\mathcal{F}}$ along the singular leaves. By Claim 2, we see that $\mathcal{L}$ admits an affine structure as a lamination in $\hat{N}$. Since $B \cap B_+$ ($B \cap B_-$ resp.) is a pinching of $D_1^+, \dots D_m^+$ ($D_1^-, \dots . D_n^-$ resp.), we see that $\mathcal{L}$ is fully carried by B.

This completes the proof of the latter half of Theorem 5.4.

§6 Examples

In this section, we use the notations L, E, $S_\pm$, $B_\pm$, B and $\tau_\pm$ as in Section 2.

$\boxed{Figure\ 6.1a}$, $\boxed{Figure\ 6.1bcde}$, $\boxed{Figure\ 6.1f}$

Example 6.1. Let L be the figure eight knot, and E an alternating diagram of L as in Figure 6.1 (a). In the following, we show that $E(L)$ contains an essential branched surface which fully carries an affine lamination. We note that the lamination is actually obtained from a stable lamination of the pseudo-Anosov monodromy of the surface bundle structure on $E(L)$ by taking a mapping torus, and hence does not admit a non-trivial transverse measure. For a proof of this fact, see Appendix D.

By Figure 6.1 (b) and (c), we see that there exists a branched surface B in $E(L)$ which is in standard position with respect to E such that the systems of generating disks for $B \cap B_+$ ($B \cap B_-$ resp.) consists of three disks D_1^+, D_2^+, D_3^+ (D_1^-, D_2^-, D_3^- resp.) as in Figure 6.1 (d)(Figure 6.1 (e) resp.), where we denote $\partial D_i^\pm$ by $\ell_i^\pm$. To be precise, the disks are pinched as follows to yield B; $D_1^\pm, D_2^\pm$ and $D_3^\pm$ are mutually parallel in $B_\pm$, and $D_1^\pm$ and $D_2^\pm$ ($D_2^\pm$ and $D_3^\pm$ resp.) are pinched to give rise to a branch locus $\alpha_\pm$ ($\beta_\pm$ resp.) in $B \cap B_\pm$, where α_+ and β_+ (α_- and β_- resp.) intersect in one point and $\alpha_+ \cup \alpha_-$ and $\beta_+ \cup \beta_-$ are two branch loci of B.

It is a routine work to see that B satisfies the six conditions of nice branched surface in section 4, and hence B satisfies the conditions (1), (2) and (3) of the definition of essential branched surface by Theorem 4.1.

Now we apply Method 1 in Appendix B to show that B has no disk of contact, is Reebless and does not carry a closed surface. In particular, B satisfies the condition (4) of the definition of essential branched surface.

It is directly observed in Figure 6.1 (b) that B has exactly two sectors S_a and S_b with two mutually intersecting branch loci $\alpha_+ \cup \alpha_-$ and $\beta_+ \cup \beta_-$.

Let s_i ($i = 1, \cdots, 4$) be the switches of $\tau_\pm$ as in Figure 6.1 (c), i.e., s_1 and s_2 (s_3 and s_4 resp.) correspond to $\partial\alpha_+ = \partial\alpha_-$ ($\partial\beta_+ = \partial\beta_-$ resp.) Assign weights w_a and w_b to the sectors S_a and S_b. Then by considering the switch condition in a neighborhood of each s_i, we have the following system of equations;

$$\begin{cases} w_a + w_b = w_b \\ w_a + w_b = w_a \\ w_a + w_b = w_b \\ w_a + w_b = w_a \end{cases}$$

This system of equations can only have the trivial solution $w_a = w_b = 0$. Hence by Method 1, we see that B does not carry a closed surface and is Reebless.

It is already proved by Theorem 4.1 that B does not have a disk of contact, but we also give another proof of this by using Method 1 in Appendix B. Suppose B has a disk of contact, i.e., branch locus $\alpha_+ \cup \alpha_-$ or $\beta_+ \cup \beta_-$ spans a disk of contact. In these cases, we respectively have the following systems of equations;

$$\begin{cases} w_a + w_b + 1 = w_b \\ w_a + w_b + 1 = w_a \\ w_a + w_b = w_b \\ w_a + w_b = w_a \end{cases}, \qquad \begin{cases} w_a + w_b = w_b \\ w_a + w_b = w_a \\ w_a + w_b + 1 = w_b \\ w_a + w_b + 1 = w_a \end{cases}$$

It is easy to see that both systems of equations do not have any solution. Hence by Method 1, we see that B does not have a disk of contact.

Finally, we show by using Theorem 5.3 that B fully carries an affine lamination. (Note that $B \cap B_+$, $B \cap B_-$ are simply connected.) Let $\mathbf{b}_i^+$ $(i = 1, 2, 3)$ ($\mathbf{b}_j^-$ $(j = 1, 2, 3)$ resp.) be the system of admissible weights on τ_+ (τ_- resp.) representing ℓ_i^+ (ℓ_j^- resp.). Let α_i^+, α_j^- $(i, j = 1, 2, 3)$ be positive real numbers and we put

$$\mathbf{w}^+ = \alpha_1^+ \mathbf{b}_1^+ + \alpha_2^+ \mathbf{b}_2^+ + \alpha_3^+ \mathbf{b}_3^+,$$

$$\mathbf{w}^- = \alpha_1^- \mathbf{b}_1^- + \alpha_2^- \mathbf{b}_2^- + \alpha_3^- \mathbf{b}_3^-.$$

Let F_1, F_2, F_3, F_4, F_5, F_6 be the components of $\tau_0 (= \tau_\pm \cap S_0)$ as in Figure 6.1 (f). Suppose that $\mathbf{w}^+$, $\mathbf{w}^-$ are projectively attachable along τ_0. Then on F_1 we have the following equation.

$$\frac{\alpha_3^+}{\alpha_2^-} = \frac{\alpha_1^+ + \alpha_2^+}{\alpha_3^-} = \frac{\alpha_1^+ + \alpha_2^+ + \alpha_3^+}{\alpha_2^- + \alpha_3^-}.$$

Here we note that the second equality follows from the first equality, and hence it is enough to consider the first one. It is directly seen that the same phenomena hold for the equations obtained from F_2, F_3 and F_4. Then we have the following system of equations.

$$\begin{cases} \dfrac{\alpha_3^+}{\alpha_2^-} = \dfrac{\alpha_1^+ + \alpha_2^+}{\alpha_3^-} \\ \dfrac{\alpha_3^+}{\alpha_1^- + \alpha_2^-} = \dfrac{\alpha_2^+}{\alpha_3^-} \\ \dfrac{\alpha_1^+}{\alpha_2^-} = \dfrac{\alpha_2^+ + \alpha_3^+}{\alpha_1^-} \\ \dfrac{\alpha_1^+}{\alpha_2^- + \alpha_3^-} = \dfrac{\alpha_2^+}{\alpha_1^-}. \end{cases}$$

Here we note that since F_5 (F_6 resp.) consists of one edge, it is obvious that $\mathbf{w}^+$ and $\mathbf{w}^-$ are projectively equivalent on F_5 and F_6 for any positive α_i^+, α_j^- $(i, j = 1, 2, 3)$.

It is easy to see that the non-trivial positive solutions of the above system is of the following form.

$$(\alpha_1^+, \alpha_2^+, \alpha_3^+, \alpha_1^-, \alpha_2^-, \alpha_3^-) = (\frac{1+\sqrt{5}}{2}c, c, \frac{1+\sqrt{5}}{2}c, \frac{1+\sqrt{5}}{2}d, d, \frac{1+\sqrt{5}}{2}d),$$

where c, d are arbitrarily fixed positive real numbers. Hence, by Theorem 5.3, we see that B fully carries a lamination which is affine in $N(B)$. Note that for any pair of positive numbers c, d, the resulting affine structures are projectively isomorphic. This fact can be confirmed as in the following. Recall the construction of the affine structure in "Proof of if part of Theorem 5.3. "That is, we first construct a foliation $\mathcal{F}_+$ ($\mathcal{F}_-$ resp.) on N_+ (N_- resp.) with transverse invariant measure corresponding to
$(\frac{1+\sqrt{5}}{2}c, c, \frac{1+\sqrt{5}}{2}c)$ $((\frac{1+\sqrt{5}}{2}d, d, \frac{1+\sqrt{5}}{2}d)$ resp.)

Since B_+, B_- are connected, and simply connected, the manifold, say N_6, obtained from N_+ and N_- by pasting them along f_6 is simply connected. Hence according to the construction, we multiply the transverse measure on $\mathcal{F}_-$ by a constant number (in fact, this is c/d) and give a transverse measure on the foliation $\mathcal{F}_+ \cup \mathcal{F}_-$ in N_6. The construction shows that this measure is used to give the affine structure. Here we note that the multiplication by c/d results in

$$(\alpha_1^+, \alpha_2^+, \alpha_3^+, \alpha_1^-, \alpha_2^-, \alpha_3^-) = (\frac{1+\sqrt{5}}{2}c, c, \frac{1+\sqrt{5}}{2}c, \frac{1+\sqrt{5}}{2}c, c, \frac{1+\sqrt{5}}{2}c).$$

This shows that the resulting affine structures are projectively isomorphic.

Example 6.2. Let L be the knot 6_1 of Rolfsen's table [Ro], and E an alternating diagram of L as in Figure 6.2 (a). We here give an example of a branched surface which carries non projectively-isomorphic one-parameter family of affine laminations.

$\boxed{Figure\ 6.2a}$, $\boxed{Figure\ 6.2bcde}$, $\boxed{Figure\ 6.2f}$

By Figure 6.2 (b), we see that there exists a branched surface B in $E(L)$ which is in standard position with respect to E such that the generating system of disks for $B \cap B_+$ ($B \cap B_-$ resp.) consists of disks D_1^+, D_2^+, D_3^+, D_4^+ (D_1^-, D_2^-, D_3^- resp.), where $\ell_i^{\pm} = \partial D_i^{\pm}$ appears as in Figure 6.2 (d), (e) and the branch loci of $B \cap B_{\pm}$ consist of pairwise disjoint arcs. We note that B is isotopic to one as obtained in [B3] and hence is essential in $E(L)$. For a proof of this fact, see Appendix E. However, we also note that B does not satisfy the condition (1) of nice branched surface in section 4 (indeed, τ_+ is not connected). This shows that the conditions of Theorem 4.1 are too strong for branched surfaces to be essential. We anyway show that B has no disk of contact, is Reebless and carries no closed surface by using Method 1 in Appendix B. Actually it is easily confirmed that B has only one sector. Hence by Fact 1 in Section 4, we obtain the above conclusion.

Finally we show by using Theorem 5.3, that B fully carries an affine lamination. Let $\mathbf{b}_i^+$ $(i = 1, 2, 3, 4)$ ($\mathbf{b}_j^-$ $(j = 1, 2, 3)$ resp.) be the system of admissible weights on τ_+ (τ_- resp.) representing ℓ_i^+ (ℓ_j^- resp.). Let α_i^+ $(i = 1, 2, 3, 4)$, α_j^- $(i, j = 1, 2, 3)$ be positive real numbers and we put

$$\mathbf{w}^+ = \alpha_1^+ \mathbf{b}_1^+ + \alpha_2^+ \mathbf{b}_2^+ + \alpha_3^+ \mathbf{b}_3^+ + \alpha_4^+ \mathbf{b}_4^+,$$

$$\mathbf{w}^- = \alpha_1^- \mathbf{b}_1^- + \alpha_2^- \mathbf{b}_2^- + \alpha_3^- \mathbf{b}_3^-.$$

Let F_1, F_2, F_3, F_4 be the components of $\tau_0 (= \tau_{\pm} \cap S_0)$ as in Figure 6.2 (f). Suppose that $\mathbf{w}^+$, $\mathbf{w}^-$ are projectively attachable along τ_0. Then, as in Example 6.1, it is enough to consider the following system of equations obtained from F_1, F_2, F_3, F_4.

$$\begin{cases} \dfrac{\alpha_2^+}{\alpha_2^-} = \dfrac{\alpha_1^+}{\alpha_1^-} \\ \dfrac{\alpha_1^+}{\alpha_2^-} = \dfrac{\alpha_2^+}{\alpha_3^-} \\ \dfrac{\alpha_3^+}{\alpha_1^-} = \dfrac{\alpha_4^+}{\alpha_2^-} \\ \dfrac{\alpha_3^+}{\alpha_2^-} = \dfrac{\alpha_4^+}{\alpha_3^-}. \end{cases}$$

We consider the following 1-parameter family of solutions of the above equations.

$$\mathbf{w}(t) = (\alpha_1^+, \alpha_2^+, \alpha_3^+, \alpha_4^+, \alpha_1^-, \alpha_2^-, \alpha_3^-) = (t, 1, t, 1, t, 1, 1/t)$$

We note that if $t \neq t'$, then $\mathbf{w}(t)$ and $\mathbf{w}(t')$ restricted on $B \cap B_-$ are not projectively equivalent. Hence we see that $\mathbf{w}(t)$ and $\mathbf{w}(t')$ give projectively different transverse invariant measures on the universal cover $\tilde{B}$. This shows that B fully carries mutually non projectively-isomorphic one-parameter family of affine laminations.

Appendix A

Proposition A. *Let B be a branched surface in a 3-manifold M. Then B carries a Reeb lamination if and only if there is a Reeb branched surface carried by B.*

Proof. The proof of "if"part is clear. Hence we give a proof of "only if"part. Suppose that there is a Reeb lamination $\mathcal{L}_R$ carried by B. Without loss of generality, we may suppose that $\mathcal{L}_R$ consists of two leaves, T and $\mathcal{R}$, where T is a torus leaf and $\mathcal{R}$ is a non-compact leaf homeomorphic to $\mathbf{R}^2$. Let V be the solid torus bounded by T such that $V \supset \mathcal{R}$. We may suppose that $T \subset \text{Int } N(B)$. Then we can take a meridian disk D of V such that there exists a sufficiently small regular neighborhood $N(\partial D)$ of ∂D in D such that $N(\partial D)$ is contained in $N(B)$, that $N(\partial D)$ intersects $\mathcal{R}$ transversely, and $N(\partial D)$ is a union of subintervals of I-fibers of $N(B)$. We note that $N(\partial D) \cap \mathcal{R}$ consists of simple closed curves which are essential in $N(\partial D)$. Let A be an annulus in $N(\partial D)$ such that $A \cap \mathcal{R} = \partial A$, and let $\mathcal{A}$ be the annulus in $\mathcal{R}$ such that $\partial \mathcal{A} = \partial A$. Since $N(\partial D)$ is sufficiently small, we see that $\mathcal{A} \cap D = \partial \mathcal{A}$, and $T \cup (A \cup \mathcal{A})$ bounds a 3-manifold, say N, homeomorphic to $T \times I$, which is contained in $N(B)$. Then it is easy to see that $p(N \cup \mathcal{R})$ is a Reeb branched surface carried by B, where $p : N(B) \to B$ is the natural projection. □

Appendix B(Non existence of disk of contact, and Reeb branched surface)

Here, we discuss some methods for proving that B does not carry a disk of contact, a Reeb branched surface or a boundary parallel torus.

Method 1. We note that the following idea was used by the first author in [B].

Recall that if the system of the switch equations for B does not have any non-trivial solution, then B does not carry a compact surface properly embedded in M, and hence B carries neither a boundary parallel torus nor a Reeb branched surface. Remember that a Reeb branched surface has a sector which forms a smooth torus.

In fact, in (2) of the Proof of Theorem in [B], the following is proved.

Fact 1. *If $B-$(the branch loci) is connected (, i.e., B consists of exactly one sector), then any system of equations obtained as above does not have a non-negative integer solution, hence, B does not carry a compact surface.*

Similar arguments work for disks of contact as below.

Let C be a branch locus of B, and A_C the component of $\partial_v N(B)$ corresponding to C. Note that A_C is an annulus. Then we modify the switch equations at C as follows.

> Recall that the branch loci of B is an immersed 1-manifold with finitely many transverse self intersection. Then we remove the intersection points from the branch locus C to obtain a system of mutually disjoint 1-manifolds in M. Let ρ be one of them, and p a point in Int (ρ). Then there is a regular neighborhood D_p of p such that $D_p \cap \rho$ is an arc properly embedded in D_p and that $B \cap D_p$ consists of three half-disks, say Δ_1, Δ_2, Δ_3, with sharing $D_p \cap \rho$ as their diameters. Here we may suppose that $\Delta_1 \cup \Delta_2$ and $\Delta_1 \cup \Delta_3$ are smooth disks. Let S_i, S_j, S_k be the sectors which contains Δ_1, Δ_2, Δ_3 respectively. (Note that two or three of S_i, S_j, S_k might coincide.)
>
> We consider the following equation.
>
> $$w_i = w_j + w_k + 1.$$

See Figure B1. Now we obtain a new system of equations for weights. In Appendix B, we call this system the *second system of equation* associated to C. Let $\mathcal{X}$ be the set of the systems of weights satisfying the second system of equations associated to C. Let $\mathcal{F}$ be the set of the fiber preserving isotopy classes of disjoint unions of surfaces carried by B such that precisely one component has a single boundary loop which forms a core circle of A_C. Obviously there is a 1 to 1 correspondence between $\mathcal{X}$ and $\mathcal{F}$. Hence non-existence of solutions for the second system of equations associated to C implies non-existence of a disk of contact.

$\boxed{\textit{Figure } B1}$

Method 2. Perhaps the following is well-known to experts (see Remark 1.3 1) of [GO]).

Fact 2. *Suppose that B is a closed branched surface such that cl$(M - N(B))$ is irreducible, $\partial_h N(B)$ is incompressible in cl$(M - N(B))$, and that there is a lamination fully carried by B. If B carries a Reeb branched surface, then either B contains a disk of* *contact, or a component* X *of* $M - N(B)$ *is (disk)*$\times I$ *with* $X \cap \partial_v N(B) = \partial$*(disk)* $\times I$*, and* $X \cap \partial_h N(B) =$ *(disk)* $\times \partial I$*,*

Proof. Suppose that B carries a Reeb branched surface. Then B also carries a Reeb lamination $\mathcal{L}_R = T \cup \mathcal{R}$, where T is a torus, and $\mathcal{R}$ is a union of non-compact leaves. Without loss of generality, we may suppose that $\mathcal{R}$ consists of one leaf. Let V be the solid torus in M bounded by T such that $V \supset \mathcal{R}$. Let B' be the subset of M obtained from B by removing all sectors which do not carry $\mathcal{L}_R$. It is easy to see that B' is a branched surface such that $\mathcal{L}_R$ is fully carried by B'. Then we may suppose (by splitting $\mathcal{L}_R$ if necessary) that $\partial_h N(B') \subset \mathcal{L}_R$, and hence $\partial(\partial_h N(B')) \subset \mathcal{L}_R$. Since every point of T is an accumulation point of an infinite sequence of points of $\mathcal{R}$, any sector of B' carries $\mathcal{R}$ if it carries T. Since such sectors carry at least two portions of leaves, we do not need to split the toral leaf T. Let $\mathcal{R}'$ be the union of non-compact leaves obtained from $\mathcal{R}$ by a possible splitting operation.

Claim. *There is a component of* $\partial(\partial_h N(B'))$ *which is contained in* $\mathcal{R}'$*.*

Proof. Assume for a contradiction that no component of $\partial(\partial_h N(B'))$ is contained in $\mathcal{R}'$. Then $\partial_h N(B') \subset T$ and $\mathcal{R}' \subset \text{Int } N(B')$.

Subclaim. $V \subset N(B')$*.*

Proof. Suppose for a contradiction $\text{Int } V \cap (M - N(B')) \neq \emptyset$. Since $\mathcal{R}'$ is carries by B', we see that $\text{Int } V \cap \text{Int } N(B') \neq \emptyset$. Since Int V is arcwise connected, we can take a path α which joins a point in $\text{Int } V \cap \text{Int } N(B')$ to a point in $\text{Int } V \cap (M - N(B'))$. Since $\partial N(B')$ is separating in M, we see that $\partial N(B') \cap (\text{Int } \alpha) \neq \emptyset$. Since $\partial_h N(B') \subset T = \partial V$ and $\partial N(B') = \partial_v N(B') \cup \partial_h N(B')$, we see that $\text{Int } \alpha \cap \text{Int } \partial_v N(B') \neq \emptyset$, hence that $\text{Int } V \cap \text{Int } \partial_v N(B') \neq \emptyset$. Let Q be the component of $V \cap \partial_v N(B')$ which contains a point of $\text{Int } \alpha \cap \text{Int } \partial_v N(B')$. Since V is a closed set, we see that Q is also a closed set. We note that $\partial Q \subset T = \partial V$. Since $\text{Int } \partial_v N(B') \cap T = \emptyset$, we see that Q is a component of $\partial_v N(B')$, where $(Q \cap \partial_h N(B)) \subset T$. Since $\mathcal{R}'(\subset \text{Int } V)$ accumulates to T and $\text{Int } Q \subset \text{Int } V$, we see that $\mathcal{R}' \cap Q \neq \emptyset$, contradicting the fact that $\mathcal{R}' \cap \partial_v N(B') \neq \emptyset$.

By Subclaim, we see that the solid torus V is embedded in the I-bundle $N(B')$, where $\partial V = T$ is transverse to the fibers. This implies that V admits an I-bundle structure such that $\partial V = T$ is transverse to the fibers. However this is impossible since the base space of the I-bundle is a closed surface which is a deformation retract of V, and since V does not have a homotopy type of a closed surface. This completes the proof of Claim.

By Claim, we can take a component ℓ of $\partial(\partial_h N(B'))$ which is innermost in $\mathcal{R}'$, and let Δ' be the disk in $\mathcal{R}'$ bounded by ℓ. If $\text{Int } \Delta' \subset \text{Int } N(B')$, then a small isotopy of

Δ' gives a disk of contact for $N(B')$. It is easy to see that this disk survives when we recover $N(B)$ from $N(B')$, and this shows that there is a disk of contact in $N(B)$. If Int $\Delta' \not\subset$ Int $N(B')$, then Δ' is a component of $\partial_h N(B')$. We can recover B from $N(B')$ by attaching the removed sectors and collapsing the I-fibers of $N(B')$ to points. Since $\partial_v N(B')$ is disjoint from the vertical boundary of $N(B)$ incident to the attached sectors, a small neighborhood of $\partial\Delta'$ in Δ', denoted by $N(\partial\Delta', \Delta')$, survives in $\partial_h N(B)$.

Let Δ be the component of $\partial_h N(B)$ such that $\Delta \supset N(\partial\Delta', \Delta')$. Then a component of $\partial\Delta$ is $\partial\Delta'$, and hence the component of $\partial\Delta$ is contractible in $N(B)$.

Suppose that Δ is a disk. Since $\partial_h N(B)$ is incompressible in $\mathrm{cl}(M - N(B))$, and since $\mathrm{cl}(M - N(B))$ is irreducible, we see that the component of $\mathrm{cl}(M - N(B))$ containing Δ is of the form (disk) $\times I$. Suppose that Δ is not a disk, i.e., $\pi_1(\Delta) \neq \{1\}$. Then Δ is compressible in $N(B)$. Now we apply the argument of the proof of Proposition 4.5 of [GO]. That is, we first recall that B fully carries a lamination, say λ. Then, by splitting finitely many leaves of λ if necessary, we may suppose that $\lambda \supset \partial_h N(B)$. Then $N(B) - \lambda$ has a structure of an open I-bundle. Since an I-bundle over a surface does not admit an essential disk, we can deform the compressing disk for Δ by an isotopy relative to the boundary to a disk, say E, contained in λ. Since E is obtained from a compressing disk, we see that (Int E)$\cap\partial(\partial_h N(B)) \neq \emptyset$. Let ℓ_E be a component of (Int E)$\cap\partial(\partial_h N(B))$ which is innermost in E, and Δ_E the disk in E bounded by ℓ_E. Then, by the above arguments, we see that either Δ_E represents a disk of contact (if Int $\Delta_E \subset$ Int $N(B)$) or a component of $M -$ Int $N(B)$, say X, is of the form (disk)$\times I$ with $X \cap \partial_v N(B) = \partial(\text{disk}) \times I$ (if $\Delta_E \subset \partial_h N(B)$).

Method 3. We note that the following fact is used in the proof of Lemma 4.3 of [GO] and the proof of it is not given there. The fact implies that it is enough to check finitely many systems of admissible weights to find a torus bounding a Reeb branched surface in a given branched surface.

Fact 3. *Let B be a branched surface in a 3-manifold M. Suppose that B has no disk of contact and that there exists a torus which bounds a Reeb branched surface carried by B. Let T be such a torus and V the solid torus such that $\partial V = T$ and that V contains the Reeb branched surface. Then no I-fiber of $N(B) \cap V$ is an arc whose endpoints are contained in T. In particular, if $\mathbf{w} = (w_1, \ldots, w_k)$ is the system of admissible weights on B which represents T (hence, each w_i is a non-negative integer), then each w_i is equal to or less than 2.*

Proof. Suppose for a contradiction that there is an I-fiber, say J, of $N(B) \cap V$ such that $\partial J \subset T$. Let $\mathcal{L}_R = T \cup R$ and B' be as in the proof of above-mentioned Fact 2. That is, $\mathcal{L}_R$ is a Reeb lamination carried by the Reeb branched surface contained in V, and B' is the closed branched surface obtained from B by removing all sectors which do not carry $\mathcal{L}_R$. As in the proof of Fact 2, we may suppose that $\partial_h N(B') \subset \mathcal{L}'_R$, where $\mathcal{L}'_R = T \cup R'$ is a lamination obtained from $\mathcal{L}_R$ by applying possible splitting operations on the non-compact leaves R. Note that $N(B') \cap V$ forms an I-bundle. Since B has no disk of contact, we immediately have the following claim.

Claim 1. *Let ℓ be a component of $\partial(\partial_h N(B')) \cap R'$ which is innermost in R', and Δ the disk in R' bounded by ℓ. Then Δ is a component of $\partial_h N(B')$.*

Then we have the following.

Claim 2. *The components of $\partial(\partial_h N(B'))$ are not nested in R', i.e., there does not exist second innermost component of $\partial(\partial_h N(B'))$.*

Proof. Suppose for a contradiction that there exists a second innermost component ℓ of $\partial(\partial_h N(B')) \cap R'$. Let Δ be the disk in R' bounded by ℓ. By above-mentioned Claim 1, we see that the interior of a small neighborhood of $\partial\Delta$ in Δ is contained in Int $N(B')$. Hence by moving Δ by a small isotopy, we can obtain a disk of contact in B'. It is easy to see that the disk of contact survives in B. a contradiction.

By Claims 1 and 2, we see that each component of $\partial_h N(B') \cap \text{Int } V$ is a disk, hence each component of $V \cap \partial N(B')$ is a 2-sphere which contains exactly one component of $\partial_v N(B')$. Since V is irreducible, this gives the following.

Claim 3. *Each component of cl$(V - N(B'))$ is homeomorphic to (disk) $\times I$, where ((disk) $\times I) \cap \partial_h N(B') =$ (disk) $\times \partial I$ and ((disk) $\times I) \cap \partial_v N(B') = \partial$(disk) $\times I$.*

By Claim 3, we can extend the I-bundle structure of $N(B') \cap V$ to a codimension 2 foliation, say Σ, of the solid torus V transverse to ∂V. Recall that $\partial_h N(B') \subset \mathcal{L}'_R$. Let $\ell_1, \ldots, \ell_p$ be the non-compact leaves of $\mathcal{L}'_R$ which intersect $\partial_h N(B')$. Let $N_1, \ldots, N_q$ be the metric completions of the components of $V - (T \cup \ell_1 \cup \cdots \cup \ell_p)$. By the definition of Σ, we see that each N_i is homeomorphic to an I-bundle $\Sigma \cap N_i$, where the total space of the associated ∂I-bundle is homeomorphic to $\mathbb{R}^2$. Since there does not exist a free involution on $\mathbb{R}^2$, we see that the bundle structure on each N_i is trivial, i.e., N_i is homeomorphic to $\mathbb{R}^2 \times I$ with each $\{pt.\} \times I$ a fiber of N_i. Hence the lamination $T \cup \ell_1 \cup \cdots \cup \ell_p$ extends to a foliation, say $\mathcal{F}$, of V such that each leaf of $\mathcal{F}$ is $\mathbb{R}^2 \times \{pt.\} \subset N_i$ for some i. Since $T \cup \ell_1 \cup \cdots \cup \ell_p$ is a Reeb lamination, this shows that $\mathcal{F}$ is a Reeb foliation. We note that $\partial J \subset \partial V$ and that J is transverse to $\mathcal{F}$. However this is impossible since a Reeb lamination admits a global normal orientation.

Appendix C (Proof of Proposition 5.1)

Proposition 5.1. *Let L be a link with a diagram E, and B a closed branched surface in standard position with respect to E. Let $\mathcal{L}$ be a lamination fully carried by the branched surface $B \cap B_\pm$, which is a pinching of a system of generating disks by the definition of standard position. Then there is another system of generating disks $E_1, \dots, E_p$ for $B \cap B_\pm$ such that each leaf of $\mathcal{L}$ is isotopic to some E_i in the I-bundle $N(B \cap B_\pm)$ by a fiber preserving isotopy. For each E_i, the union of the leaves of $\mathcal{L}$ which are isotopic to E_i by a fiber preserving isotopy is a closed subset of $B_\pm$.*

In this appendix, we firstly prove:

Proposition C. *Let Z be a 3-ball, and C_0 a branched surface in Z. Suppose that C_0 is a pinching of a disjoint union of smooth disks $G_1 \cup \cdots \cup G_m$ properly embedded in Z as below.*

(1) *Each branch locus intersects ∂Z.*
(2) *No pinching occurs between subsurfaces of a single component of $G_1, \dots, G_m$, and hence the image of each G_i is a disk embedded in C_0.*

Let $\mathcal{L}$ be a lamination which is fully carried by C_0. Then C_0 is a pinching of a disjoint union of smooth disks $R_1, \dots, R_n$ properly embedded in Z such that similar conditions as (1) and (2) above hold and that each leaf of $\mathcal{L}$ is isotopic to some R_i in the I-bundle $N(C_0)$ by a fiber preserving isotopy.

Let H be a (connected) surface in ∂Z such that H is disjoint from the branch loci of C_0 and that $N(C_0) \cap H$ is a union of I-fibers of the I-bundle $N(C_0)$. (Note that $N(C_0) \cap H$ may be disconnected.) Suppose that $G_i \cap H$ consists of at most one arc properly embedded in H for every i. Then for each leaf l of $\mathcal{L}$, $l \cap H$ consists of at most one arc properly embedded in H.

Then we prove Proposition 5.1 by using Proposition C.

For the proof of Proposition C, we modify Lemma 2.5 of [GO] as in the following form.

A variation of Lemma 2.5 of [GO]**.** *Let B_* be a branched surface possibly with boundary in a 3-manifold M such that cl$(M - N(B_*))$ is irreducible. Suppose:*

(1) *B_* has no disk of contact,*
(2) *no component of cl$(M - N(B_*))$ is of the form (disk) $\times I$, where ∂(disk) $\times I \subset \partial_v N(B_0)$, and (disk) $\times \partial I \subset \partial_h N(B_0)$,*
(3) *$\partial_h N(B_*)$ is incompressible in cl$(M - N(B_*))$ and*
(4) *there are no monogons in cl$(M - N(B_*))$.*

Suppose B'_* is a splitting of B_*. Then we have:

(1) *$\partial_h N(B'_*)$ is incompressible in cl$(M - N(B'_*))$ and*

(2) *there are no monogons in cl$(M - N(B'_*))$.*

Proof of 'A variation of Lemma 2.5 of [GO]*'.* We prove only the conclusion (1). The proof of the conclusion (2) is similar, and we omit it. Since B'_* is a splitting of B_*, we have $N(B_*) = N(B'_*) \cup J$, where J is an I-bundle.

Suppose, for a contradiction, there is a compressing disk D for $\partial_h N(B'_*)$ such that $D \subset \text{cl }(M - N(B'_*))$. By standard innermost loop and outermost arc arguments, we may suppose that each component of $D \cap \partial_v J$, if exists, is either an essential simple closed curve in $\partial_v J$ or a fiber of an I-bundle structure of J.

Suppose there exists a simple closed curve component in $D \cap \partial_v J$. Then, by taking a component of $D \cap \partial_v J$ which is innermost in D, we obtain a disk D' in D such that $D' \cap \partial_v J = \partial D'$. If $D' \subset J$, then D' is a disk of contact in $N(B_*)$, contradicting (1) of the assumption. If $D' \subset \text{cl}((M - N(B_*))$, then by the condition (3) of the assumption and the irreducibility of $\text{cl}(M - N(B_*))$, we see that the component of $\text{cl}(M - N(B_*))$ containing D' is of the form (disk) $\times I$, where $\partial(\text{disk}) \times I \subset \partial_v N(B_*)$, and (disk) $\times \partial I \subset \partial_h N(B_*)$, contradicting the condition (2) of the assumption.

Suppose each component of $D \cap \partial_v J$ is a fiber of an I-bundle structure of J. There is an outermost arc of $D \cap \partial_v J$ on D, and it cuts off from D a monogon in $\text{cl}(M - N(B_*))$, contradicting the condition (4) of the assumption.

Hence $D \cap \partial_v J = \emptyset$. Then, by the condition (3) of the assumption, we see that ∂D is contractible in $\partial_h N(B_*)$, contradicting the fact that D is a compressing disk. This completes the proof of the conclusion (1).

Lemma. *The branched surface C_0 satisfies the assumptions of 'A variation of Lemma 2.5 of* [GO]*'.*

Proof. Since each branch locus of C_0 intersects ∂Z, we see that C_0 satisfies the conditions (1), (2) of the assumption of 'A variation of Lemma 2.5 of [GO]'. This also implies that each component of $\partial_h N(C_0)$ is a disk, and this shows that C_0 satisfies the condition (3). Moreover, since no pinching occurs between subsurfaces of a single component of $G_1, \ldots, G_m$, we see that C_0 satisfies the condition (4).

Proof of Proposition C.

Claim 1. *Let F be a compact 2-manifold fully carried by C_0. Then each component E of F is a disk such that*

(1) *E is mapped to an embedded disk in C_0 by the projection map $N(C_0) \to C_0$, and*

(2) *$E \cap H$ consists of at most one arc properly embedded in H.*

Proof. We note that $\partial_h N(C_0)$ has a component which is a disk properly embedded in Z, and this shows that Z is not an essential branched surface. However above-mentioned lemma shows that the branched surface C_0 satisfies the other conditions of the definition of incompressible branched surfaces. Under the conditions proved in Lemma, the arguments in [F-O] show that no component of F is a 2-sphere, and that F is incompressible in the 3-ball Z. Hence each component of F is a disk. Let E be a component of F.

Subclaim 1. *E is mapped to an embedded disk in C_0.*

Proof. By splitting some components of F if necessary, we may assume that $\partial_h N(C_0) \subset F$. Suppose for a contradiction that E is not projected to an embedded disk in C_0. Then there is an I-fiber J_0 of $N(C_0)$ which intersects the disk E at two or more points. Let J' be a subinterval of J_0 such that $\partial J' \subset E$ and $\text{int}J' \cap E = \emptyset$. If another component E' of F intersects J', then E' must intersects J' at two or more points since E and E' are disks properly embedded in the ball Z. Hence, retaking E if necessary, we can assume without loss of generality that E is the only component of F which intersects J'. Let W be the closure of a component of $N(C_0) - F$ which contains $\text{int}J'$. Since $\text{int}W$ is disjoint from F, W is an I-bundle over a subdisk of E, and E intersects $\partial_h W$. Note that $W \cap \partial Z \neq \emptyset$, otherwise W would contain an annular component of $\partial_v N(C_0)$, contradicting the assumption. Let Q' be a disk bounded by ∂E on ∂Z such that $W \cap \partial Z \subset Q'$. The I-fibers of $W \cap \partial Z$ have endpoints in ∂E. Let J be an outermost one on Q', and Q the outermost disk, that is, $Q - J$ is disjoint from W. We remove all the component of $N(C_0) - F$ which are disjoint from $\text{int}W$. This amounts to a splitting operation on the branched surface C_0, and we obtain a new branched surface C' in Z. The components of C' intersecting the disk Q are properly embedded disk in Z, and they are parallel to subdisks of Q. Hence we can isotope Q relative to its boundary so that it gives a monogon for the branched surface C'. However this contradicts the fact that C_0 does not admit a monogon, which follows from 'A variation of Lemma 2.5 of [GO]'mentioned above. This completes the proof of Subclaim 1.

Recall that H is a surface in ∂Z given in the statement of Proposition C.

Subclaim 2. *$\partial E \cap H$ consists of at most one arc properly embedded in H.*

Proof. By splitting some components of F if necessary, we may suppose that $\partial_h N(C_0) \subset F$. Suppose for a contradiction that there exists a component E of F such that $\partial E \cap H$ consists of more than one arc. Then there is a disk Q in ∂Z such that $Q \cap E = \partial Q \cap \partial E = \alpha$

an arc, and that cl $(\partial Q - \alpha) \subset H$. Then β denotes the arc cl $(\partial Q - \alpha)$. If another component E' of F intersects β, then E' must intersects β at two or more points since E and E' are disks properly embedded in the ball Z. Hence, retaking E if necessary, we can assume without loss of generality that E is the only component of F which intersects β. Note that every component of F intersecting Int Q is a disk whose boundary is entirely contained in Int Q. Hence by moving Int Q by an isotopy, we obtain a disk Q' such that $\partial Q' = \partial Q = \alpha \cup \beta$ and that Int $Q' \cap F = \emptyset$. Let $N(F)$ be a sufficiently small regular neighborhood of F, and N_F the union of the closures of several components of $N(F) - F$ such that $N_F \supset \partial_h N(C_0)$ and that $N_F \subset N(B_0)$. Note that $Q' \cap N_F = Q' \cap E (= \alpha)$. rel. β Since F is fully carried by C_0, there is an I-bundle G in Z with base space a compact 2-manifold such that $N(C_0) = N_F \cup G$, where $G \cap N(F) = G \cap \partial_h N(F) = \partial_h G$. Since each component of $\partial_v N(C_0)$ is a disk, we may suppose, by innermost loop arguments, that each component of $\partial_v N(C_0) \cap Q'$ is (if exists) a proper arc which is an I-fiber of G. Note that these arcs are properly embedded in Q'. Then we take an outermost component of $\partial_v N(C_0) \cap Q'$ on Q' with an outermost disk Δ such that $\Delta \cap \beta = \emptyset$. Note that Δ is a monogon in cl $(Z - N(C_0))$, contradicting Subclaim 1 mentioned above. Hence $\partial_v N(C_0) \cap Q' = \emptyset$. Let $Q_1, \ldots, Q_{2m}$ be duplicated parallel copies of $G_1, \ldots, G_m$ in $N(C_0)$ such that $\partial_h N(C_0) \subset Q_1 \cup \cdots \cup Q_{2m}$. Since $\partial_v N(C_0) \cap Q' = \emptyset$, there is a component of $\partial_h N(C_0)$ which contains the arc α entirely. Hence Q_k contains α entirely for some $1 \leq k \leq 2m$. Remember that the subarc β of $\partial Q''$ connects distinct components of $\partial E \cap H$. These arcs are contained also in ∂Q_k since the surface H_j is disjoint from the branch loci of C_0. Hence ∂Q_k intersects H in two or more arcs. This contradicts the assumption, and this completes the proof of Subclaim 2.

Subclaims 1 and 2 complete the proof of Claim 1.

For the definition of a foliation which is a thickening of a lamination, see Definition 2.1 of [GO]. Let $\mathcal{F}$ be a foliation on $N(C_0)$ which is a thickening of $\mathcal{L}$. Recall that C_0 is a pinching of the disks $G_1, \ldots, G_m$. We may suppose, by exchanging suffix if necessary, that G_1 is an outermost component of $G_1, \ldots, G_m$, i.e., a component of $Z - G_1$ does not intersect $G_2 \cup \cdots \cup G_m$.

Then there is a component of $\partial_h N(C_0)$, say R_1, which is parallel to G_1. By the way of construction of the thickening $\mathcal{F}$, R_1 is a leaf of $\mathcal{F}$. By Reeb stability theorem (see, for example, Lemma 2.2 of [GO]), we see that the leaves which are close to R_1 are parallel to R_1 in $N(C_0)$. Let $\mathcal{R}_1$ be the union of the leaves of $\mathcal{F}$ which are isotopic to R_1 by fiber preserving isotopies in the I-bundle $N(C_0)$. Then we see that, by Reeb stability theorem, $\mathcal{R}_1$ is homeomorphic to $R_1 \times I$ with each $R_1 \times \{p\}$ corresponding to a leaf for

$p \in [0,1]$ and with $R_1 \times \{0\}$ corresponding to R_1. Let R'_1 be the leaf of $\mathcal{F}$ corresponding to $R_1 \times \{1\}$. Then we see that $R'_1 \cap \partial_h N(C_0)$ is a non-empty union of components of $\partial_h N(C_0)$. Let $S_1, \ldots, S_k$ be the sectors of C_0 corresponding to $R'_1 \cap \partial_h N(C_0)$. Let $C_1 = c\ell(C_0 - (S_1 \cup \cdots \cup S_k))$. It is easy to see that C_1 is a branched surface.

Let $N'_1 = c\ell(N(C_0) - \mathcal{R}_1))$. Note that $N'_1 \cap \mathcal{R}_1 = c\ell(R'_1 - (R'_1 \cap \partial_h N(C_0)))$, and each component of this is a disk. Let J_1 be the union of leaves of $\mathcal{F}$ which are isotopic to a component of $c\ell(R'_1 - (R'_1 \cap \partial_h N(C_0)))$ by fiber preserving isotopies in the I-bundle $N(C_0)$. We see that J_1 is homeomorphic to $c\ell(R'_1 - (R'_1 \cap \partial_h N(C_0))) \times [0,1)$ with each $c\ell(R'_1 - (R'_1 \cap \partial_h N(C_0))) \times \{p\}$ corresponding to a union of leaves for $p \in [0,1)$ and with $c\ell(R'_1 - (R'_1 \cap \partial_h N(C_0))) \times \{0\}$ corresponding to $c\ell(R'_1 - (R'_1 \cap \partial_h N(C_0)))$. Let $N_1 = N'_1 - J_1$. We note that $\mathcal{F} \cap N_1$ is a foliation on N_1, that N_1 is a fibered neighborhood of the branched surface C_1 above, and that each fiber of N_1 intersects $\mathcal{F} \cap N_1$ transversely.

Claim 2. *There exists a system of disjoint union of disks $R_2, \ldots, R_n$ properly embedded in Z such that*

(1) *each leaf of $\mathcal{L} \cap N_1$ is isotopic to some R_i by a fiber preserving isotopy in the I-bundle N_1 and*

(2) *no pinching occurs between subsurfaces of a single component of $R_2, \ldots, R_n$, i.e., the image of each R_i is a disk embedded in C_1 and*

(3) *for the surface H in Proposition C, $R_i \cap H$ consists of at most one arc properly embedded in H for each i.*

Proof. Recall that C_0 is a pinching of the disks $G_1, \ldots, G_m$. Let $Q_1, \ldots, Q_{2m}$ be duplicated parallel copies of $G_1, \ldots, G_m$ in $N(C_0)$ such that $\partial_h N(C_0) \subset Q_1 \cup \cdots \cup Q_{2m}$. For a component ρ of $\partial(\partial_v N(C_0))$, $Q(\rho)$ denotes the component of $Q_1, \ldots, Q_{2m}$ which contains ρ. Recall that $S_1, \ldots, S_k$ are the sectors of C_0 corresponding to $R'_1 \cap \partial_h N(C_0)$. It is easy to see that $c\ell(C_0 - S_1)$ is a branched surface, say C'_1. We show that C'_1 is a pinching of disjoint union of disks each component of which satisfies similar conditions as those of of the conclusion of Claim 2. We will see repetitions of the following arguments complete the proof of Claim 2. Let $\rho_1, \ldots, \rho_p$ be the components of the frontier in R'_1 of the component of $\partial_h N(C_0)$ corresponding to S_1. Then let ρ'_i be the component of $\partial_v N(C_0) \cap \partial_h N(C_0)$ such that ρ_i and ρ'_i are contained in the same component of $\partial_v N(C_0)$. Then $Q(\rho'_i)$ is cut into two disks by ρ'_i. We denote by $Q'(\rho'_i)$ the closure of the component of $Q(\rho'_i) - \rho'_i$ such that a small neighborhood of ρ'_i in $Q'(\rho'_i)$ is contained in $\partial_h N(C_0)$. Note that $Q(\rho'_i)$ does not intersect the I-fibers of $N(B)$ which intersect the interior of the sector S_1, otherwise there would be an I-fiber of $N(B)$ intersecting ρ_i such that $Q(\rho'_i)$ intersects it at two or more points, contradicting the assumption. Let $G'_1, \ldots, G'_n$ be the components of $G_1, \ldots, G_m$ whose subdisks are carried by S_1, $G_1'', \ldots, G_r''$ the disks obtained from $G_1', \ldots, G_n'$ by removing the subdisks carried by S_1 and taking the closures.

We can naturally join the components of $G_1'', \ldots, G_r''$ and copies of $Q'(\rho_1'), \ldots, Q'(\rho_p')$ by using components of $\partial_v N(C_0)$ to obtain a system of (not necessarily mutually disjoint) surfaces carried by C_1'. Then we perturb the surfaces to be in a general position with keeping them to be carried by C_1', and apply suitable cut and paste operations to obtain a compact 2-manifold F_1' fully carried by C_1'. We note that $F_1' \cup G_1$ is fully carried by C_0. Hence by Claim 1 mentioned above, we see that each component of $F_1' \cup G_1$ is a disk satisfying similar conditions as those fo the conclusion of Claim 2. It is easy to see that the above arguments can be repeated to give the conclusion of Claim 2.

Since the number of the sectors of C_0 is finite, we see that the above argument can be repeated to give Proposition C.

Proof of Proposition 5.1. Let $H = B_\pm \cap (\text{crossing balls})$. Note that H is a disjoint union of disks in $\partial B_\pm$. Then let $H' = \mathrm{cl}(H - N(L))$, where L is the link. Then we apply Proposition C to $\mathcal{L}$ with $B_\pm$, $D_1, \ldots, D_m$ and H' regarded as Z, $G_1, \ldots, G_m$ and H respectively. Then we obtain a system of mutually disjoint disks $\{E_1, \ldots, E_p\}$ properly embedded in $B_\pm$ such that; $B_\pm$ is a pinching of $E_1 \cup \cdots \cup E_p$, where no pinching occurs between subsurfaces of a single component of $E_1, \ldots, E_p$,; each leaf of $\mathcal{L}_\pm$ is isotopic to some E_i in the I-bundle $N(B \cap B_\pm)$ by a fiber preserving isotopy, and ; the boundary of each E_i does not meet the same side of a bubble more than once. These show that $E_1, \ldots, E_p$ is a system of generating disks for $B \cap B_\pm$.

Let $\mathcal{E}_i$ be the union of leaves of $\mathcal{L}$ which are isotopic to E_i by fiber preserving isotopies in the I-bundle $N(C_0)$. Let $\mathcal{F}$ be a foliation on $N(C_0)$ which is a thickening of $\mathcal{L}$, and $\mathcal{E}_i'$ the union of the leaves of $\mathcal{F}$ which are isotopic to E_i by fiber preserving isotopies in the I-bundle $N(C_0)$. Then we have $\mathcal{E}_i = \mathcal{E}_i' \cap \mathcal{L}$. By Reeb stability theorem, $\mathcal{E}_i'$ is homeomorphic to $E_i \times I$ with each $E_i \times \{p\}$ corresponding to a leaf for $p \in [0, 1]$. Since $\mathcal{L}$ is a closed set, this shows that $\mathcal{E}_i$ is a closed subset of $B_\pm$. This completes the proof of Proposition 5.1.

Appendix D

In general, let $\psi : F \to F$ be a pseudo-Anosov homeomorphism of a surface F, and ν the stable lamination of ψ. Let M be the mapping torus of ψ, i.e., M is obtained from $F \times I$ by identifying $F \times \{1\}$ and $F \times \{0\}$ by the homeomorphism $(x, 1) \to (\psi(x), 0)$. Then we can obtain a lamination λ in M as the image of $\nu \times I$ in F. We call λ the *mapping torus of the stable lamination* ν.

In this appendix, we will show that the essential lamination of Example 6.1 is the mapping torus of a stable lamination of the pseudo-Anosov monodromy of the surface bundle structure of $E(L)$.

We will demonstrate this by a picture of the branched surface obtained from B by cutting along a minimal genus Seifert surface of the figure eight knot. For the convenience of drawing, we slightly modify the diagram E as in Figure D-1 (a), and take a minimal genus Seifert surface S drawn for E as in Figure D-1 (b). It is directly observed that $B \cap S$ is a train track τ_* as in Figure D-1 (b). Let D_1, D_2, D_3, D_4 be crossing balls for E. Then $D_i \cap B$ is a saddle-shaped disk, say R_i, in D_i as in Figure 2.1. See Figure D-1 (a).

$\boxed{Figure\ D-1}$

Here we may suppose that $R_1 \cap S$ ($R_2 \cap S$ resp.) is a diagonal line of the square R_1 (R_2 resp.) and that $R_3 \cap S = \emptyset$, $R_4 \cap S = \emptyset$. Then we may take a "straight" arc σ in B joining a vertex of R_3 and a vertex of R_4 as in Figure D-1 (a). As directly observed from Figure D-1 (a), there is a hexagon H in B such that H is obtained from $R_3 \cup \sigma \cup R_4$ by expanding σ in B and that $H \cap S$ consists of two edges of H contained in edges of τ_*. Note that among the other four edges of H two are contained in edges of τ_+ and the other two edges are contained in τ_-. See Figure D-2.

$\boxed{Figure\ D-2}$

It is also directly observed from Figure 6.1 and Figure D-2 that $\mathrm{cl}(\tau_\pm - (\tau_* \cup R_1 \cup R_2 \cup H))$ consists of two Y-shaped 1-complexes, say Y_1 and Y_2. See Figure D-2 (b). Recall that $B \cap B_+$ ($B \cap B_-$ resp.) is a union of three disks D_1^+, D_2^+, D_3^+ (D_1^-, D_2^-, D_3^- resp.) as in Figure D-3 (a) (Figure D-4 (a) resp.). It is easy to see that $B \cap B_+$ ($B \cap B_-$ resp.) is homeomorphic to the branched surface of Figure D-3 (b) (Figure D-4 (b) resp.). Let $B_\sharp$ be the branched surface obtained from B by cutting along S. Let R_i^+, R_i^- be the four triangles obtained by cutting the saddle disk R_i along the arc $\tau_* \cap R_i$ such that $R_i^+ \subset B_+$ and $R_i^- \subset B_-$. Note that the branched surface $B_\sharp$ can be obtained also from the disjoint union of $\mathrm{cl}((B \cap B_+) - H)$ and $\mathrm{cl}((B \cap B_-) - H)$ by pasting them along $Y_1 \cup Y_2 \cup \sigma$ and attaching $R_1^+, R_1^-, R_2^+, R_2^-, H$ along the subarcs of their boundaries in the bubbles. This can be done in an abstract way as in Figure D-5 to obtain the branched surface of Figure D-6. It is easy to see from Figure D-6 that any surface carried by $B_\sharp$ is homeomorphic to either $S^1 \times [0,1]$ or $\mathbb{R} \times [0,1]$, where a neighborhood of one boundary component is contained in the +-side of S and a neighborhood of the other boundary component is contained in the $-$-side of S.

Figure D − 3	*Figure D − 4*	*Figure D − 5*

Now we consider the affine lamination associated to the system of solutions

$$(\alpha_1^+, \alpha_2^+, \alpha_3^+, \alpha_1^-, \alpha_2^-, \alpha_3^-) = (\frac{\sqrt{5}+1}{2}, 1, \frac{\sqrt{5}+1}{2}, 1, \frac{\sqrt{5}-1}{2}, 1)$$

of the system of equations in Example 6.1. Note that this solution gives systems of admissible weights on $B \cap B_+$ and $B \cap B_-$ which agree at $Y_1 \cup Y_2$ and gives a system of admissible weights on the branched surface $B_\sharp$. It is directly seen that the system of admissible weights induces two systems of admissible weights on τ_* as in Figure D-7, each from boundary components of $B_\sharp$. It is easy to see that these systems of weights are projectively equivalent. In fact these weights give a stable lamination of the pseudo-Anosov monodromy of the surface bundle structure of $E(L)$. Hence the above lamination is a mapping torus of the stable lamination.

Figure D − 6a	*Figure D − 6b*	*Figure D − 7*

Appendix E

We will claim that the branched surface of Example 6.2 is an essential branched surface obtained by the first author in [B3].

The picture of Figure E-1 (a) is borrowed from Figure 3 of [B3]. Note that the diagram of a knot in Figure E-1 (a) is a non-alternating diagram of 6_1. In [B3], it is shown that the branched surface of Figure E-1 (a) is an essential branched surface in the knot complement. If we put the branched surface “tamely ” with respect to the diagram, then the intersection of the branched surface and the projection 2-sphere will look as in Figure E-1 (b). Then we move a part of the knot as the broken line in Figure E-1 (b). Then it is directly observed that the image of the branched surface by this deformation is actually the branched surface of Example 6.2.

Figure E − 1

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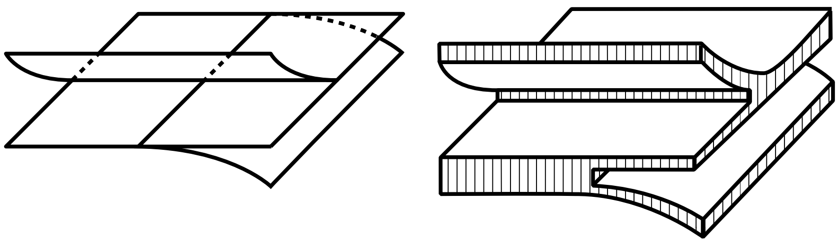

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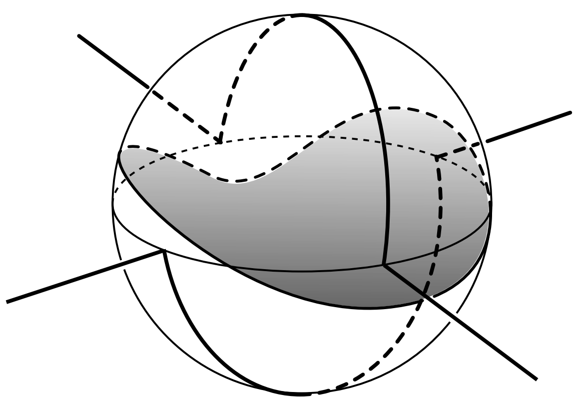
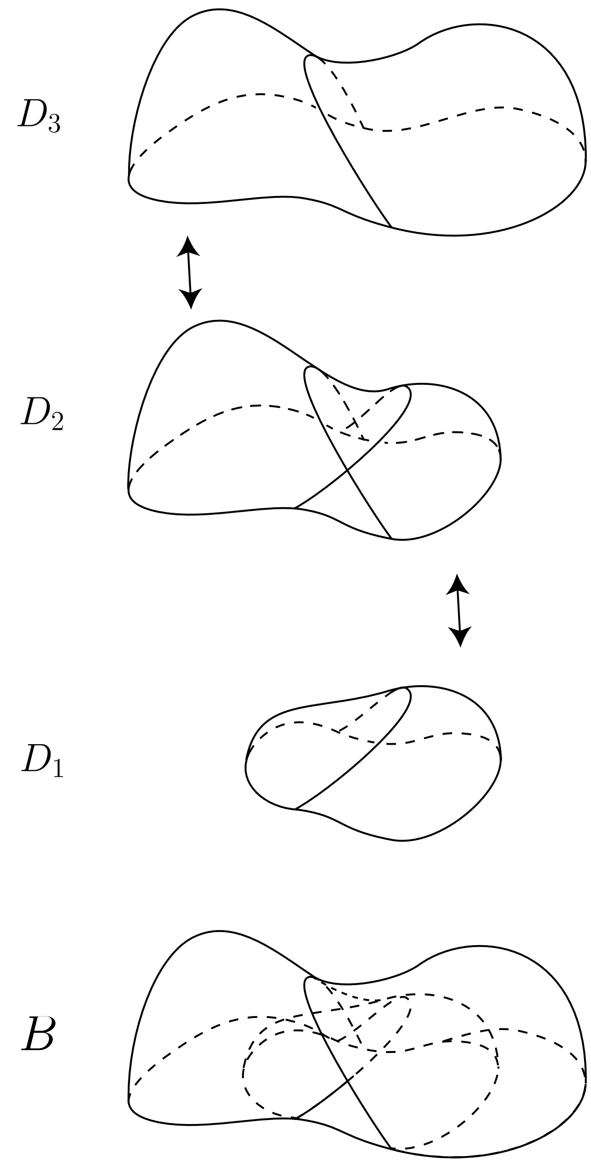


Figure 2.2



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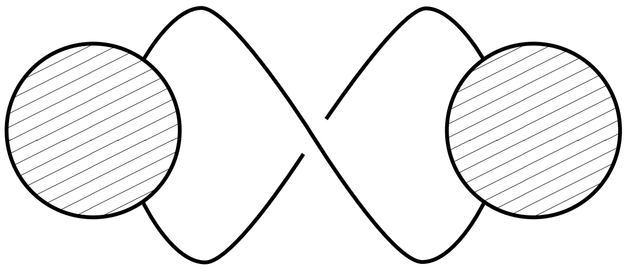

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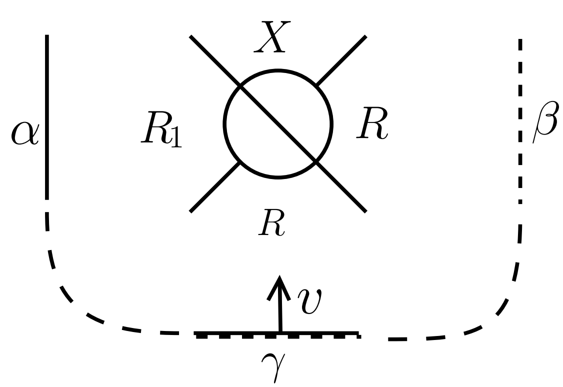

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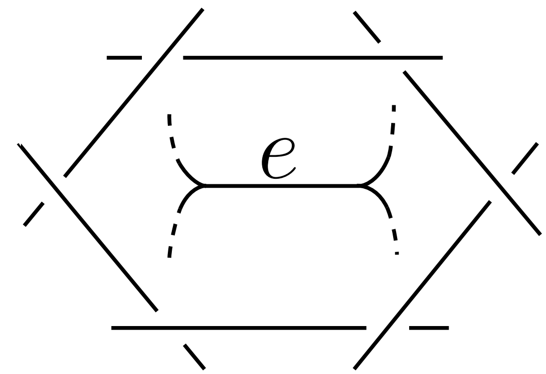

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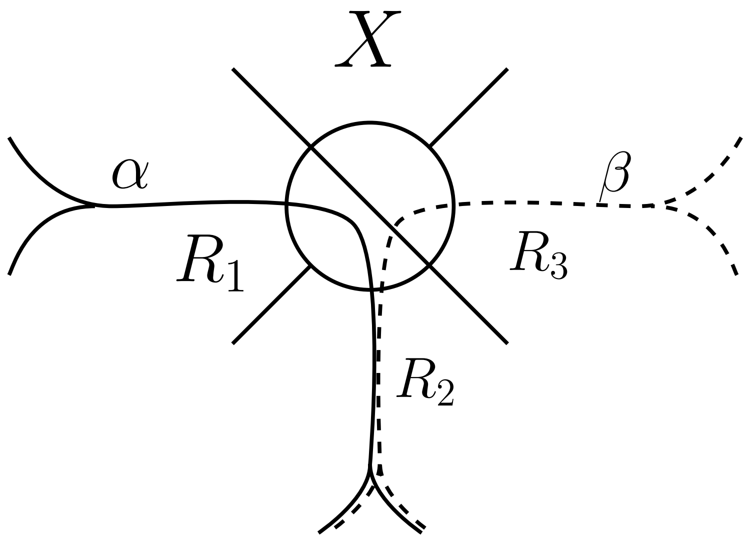

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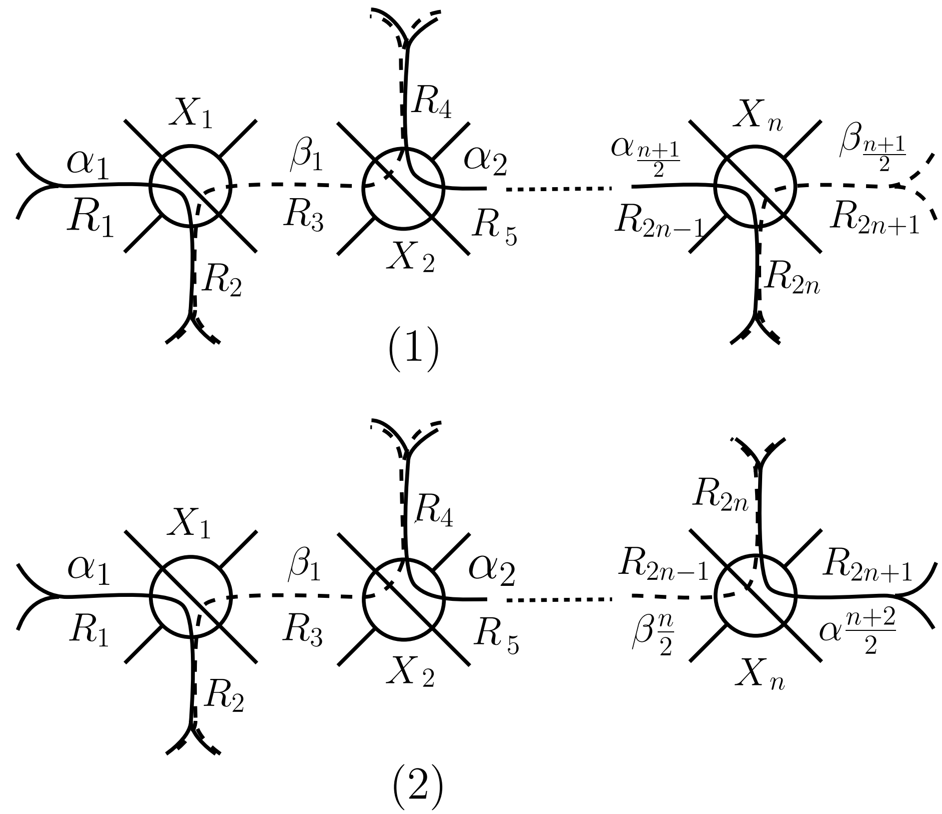

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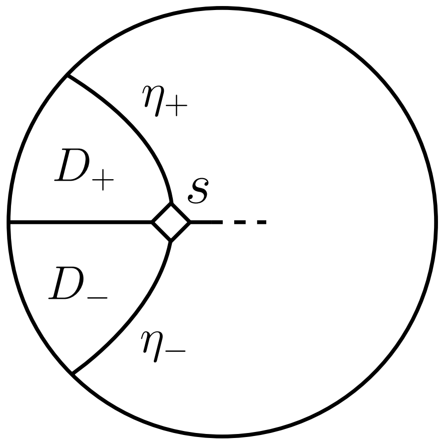

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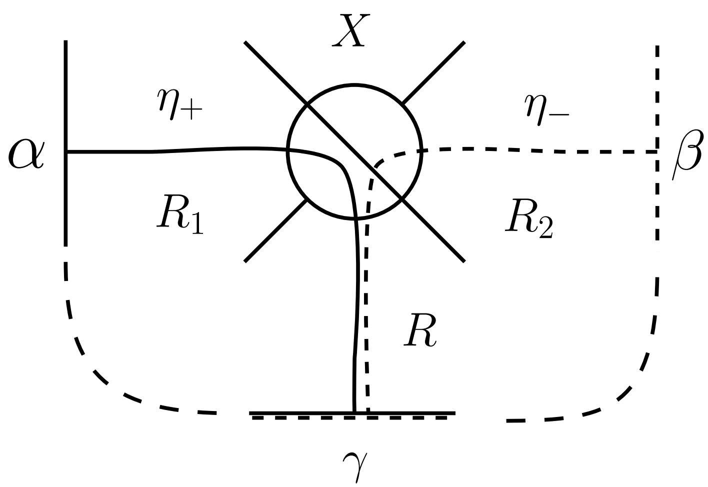

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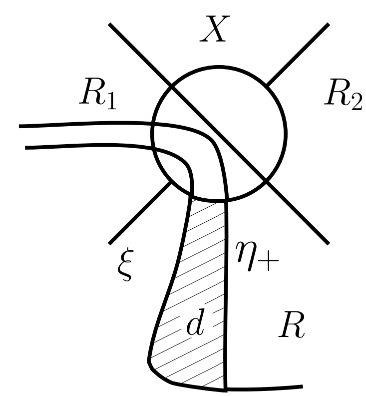

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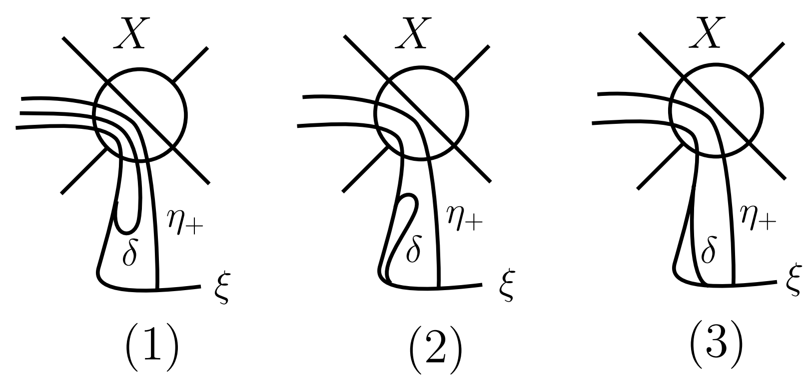

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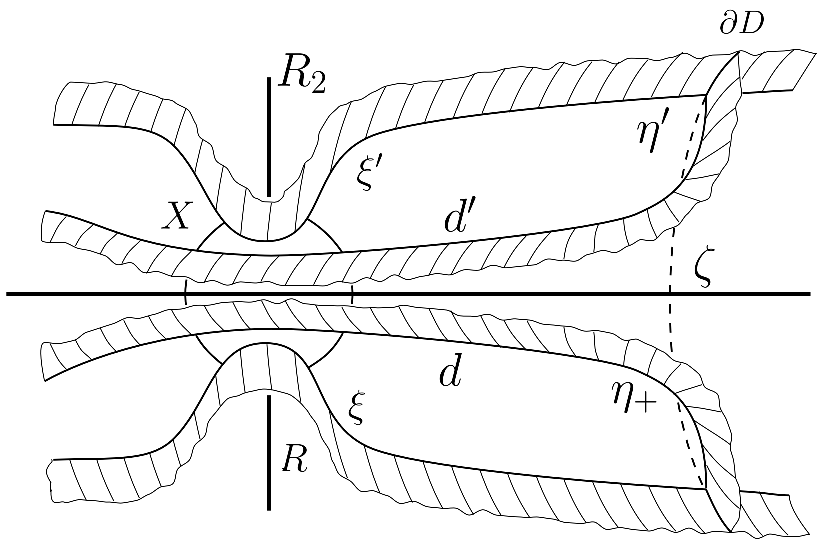

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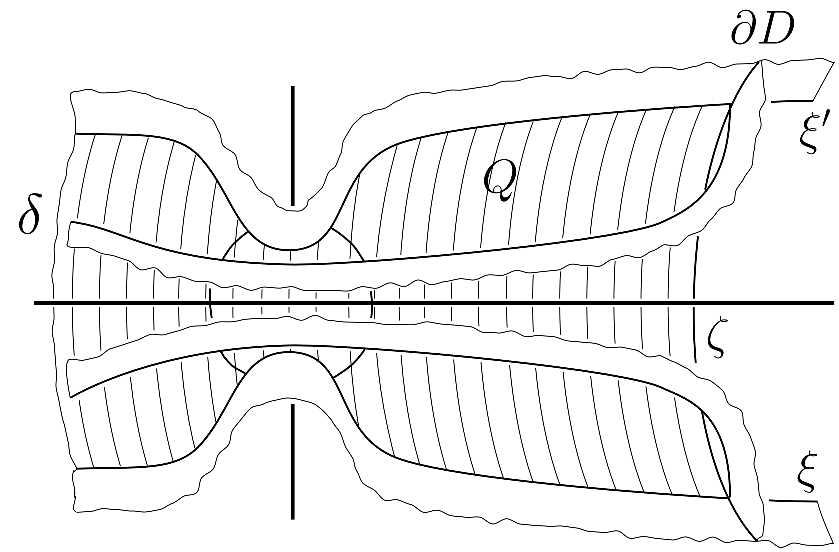

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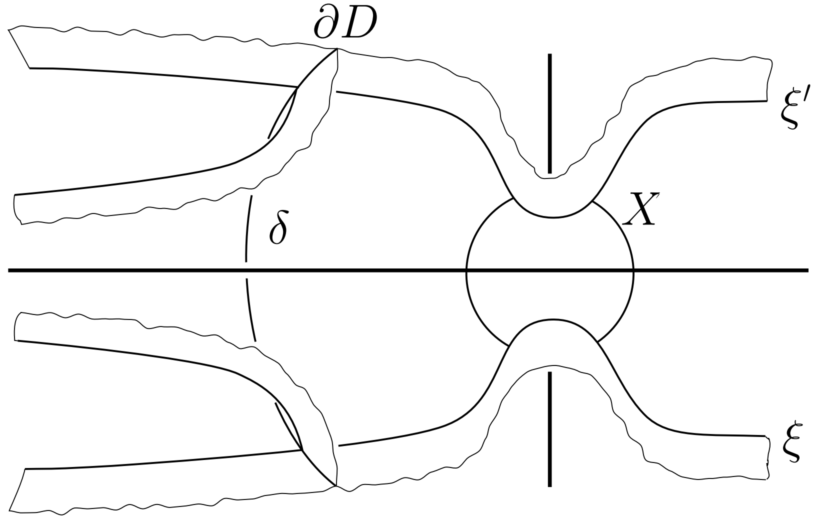

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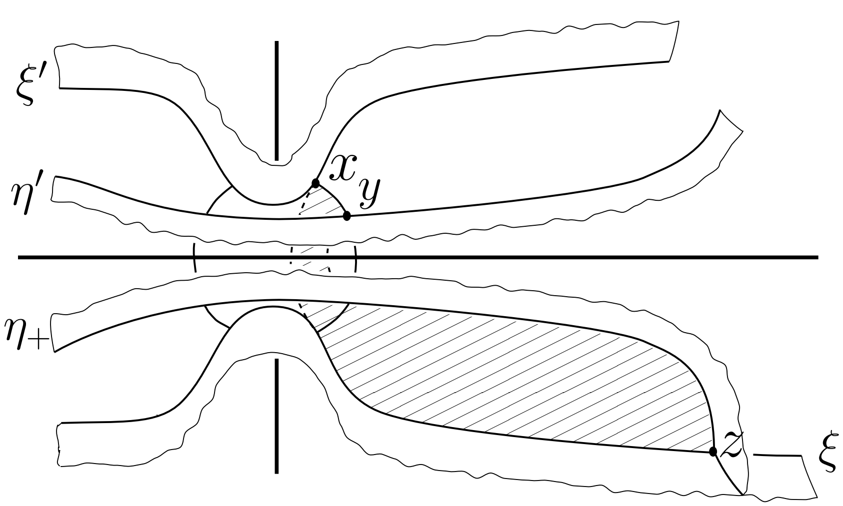

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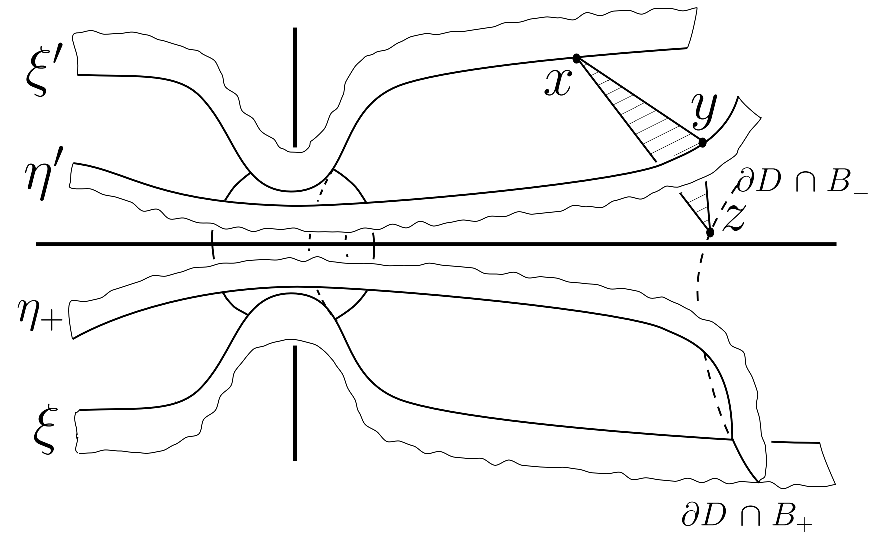

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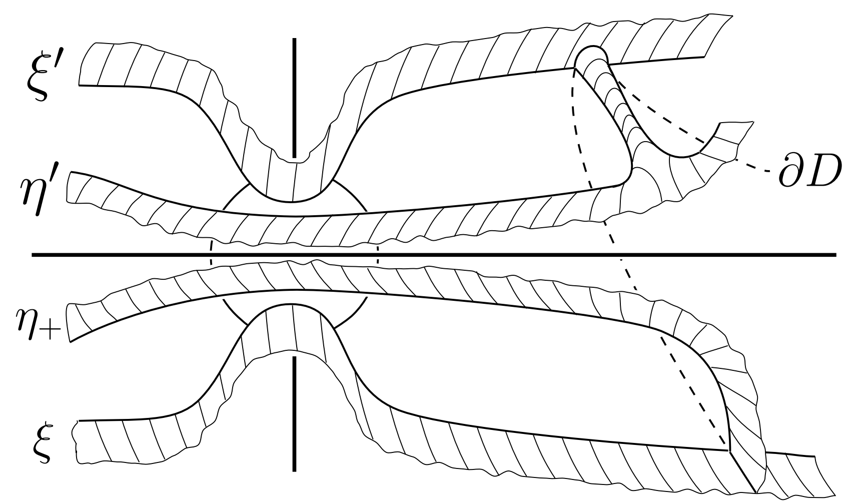

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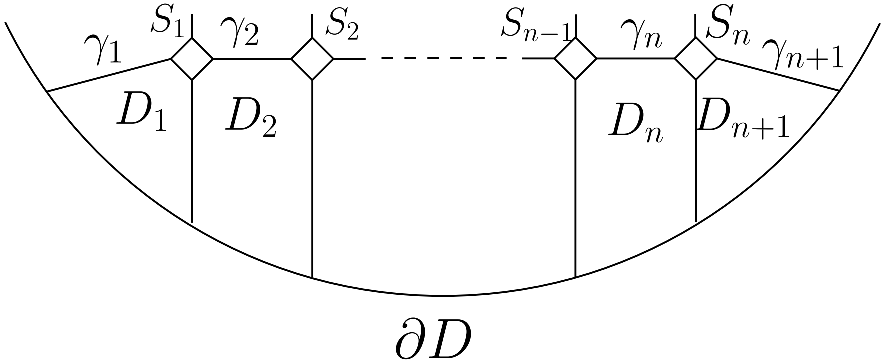

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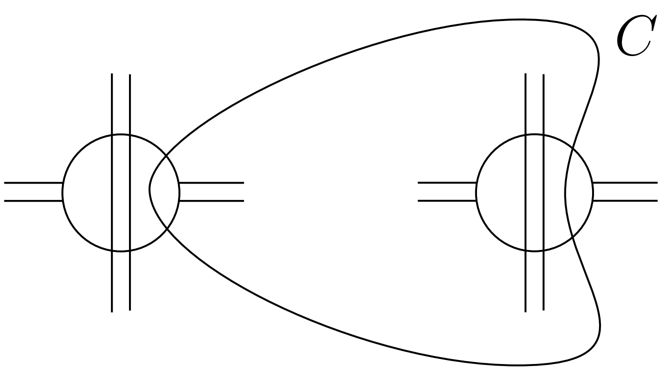

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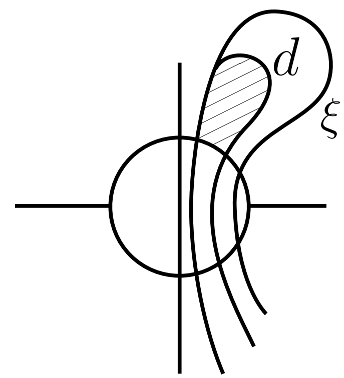

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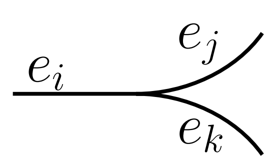

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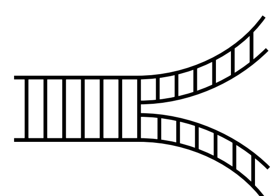

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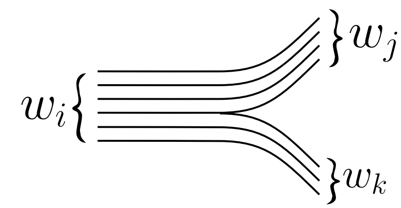

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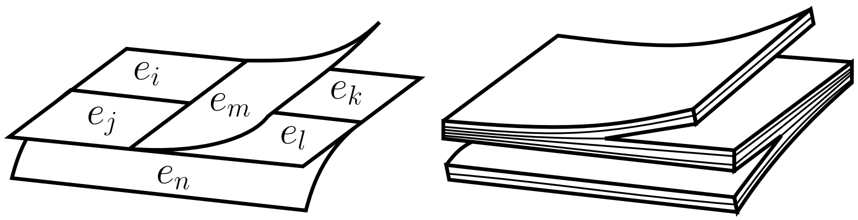

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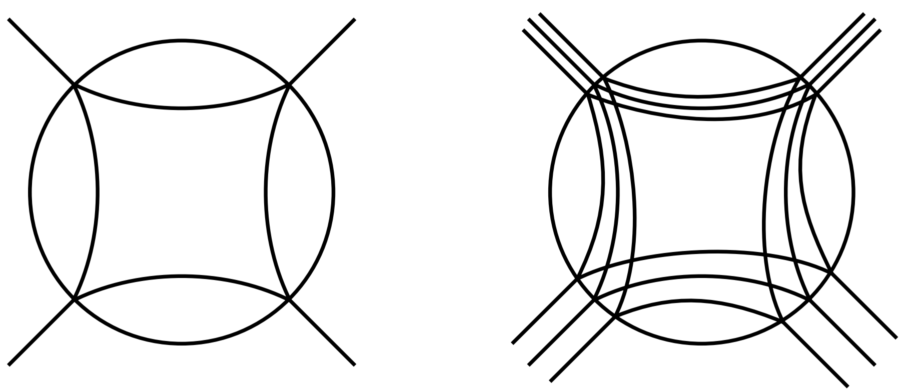

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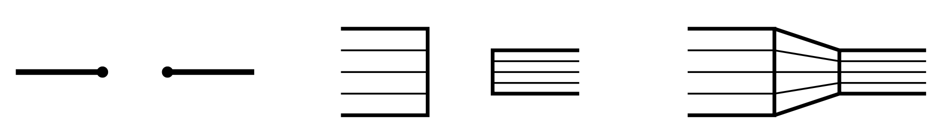

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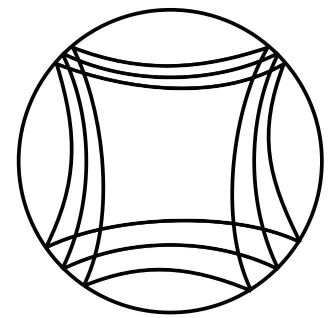

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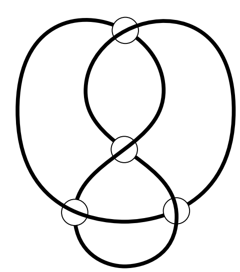
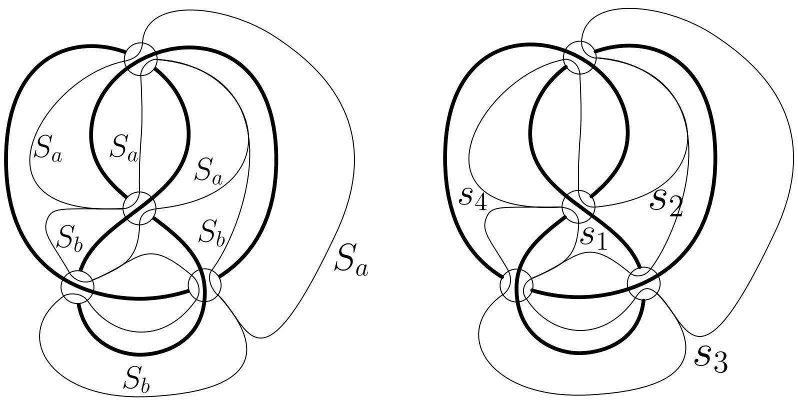
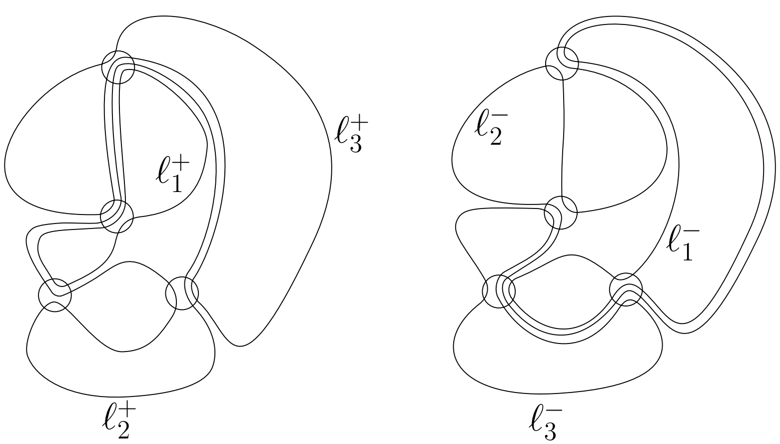


Figure 6.1 (a)



τ_+ τ_-

(b) (c)



(d) (e)

Figure 6.1

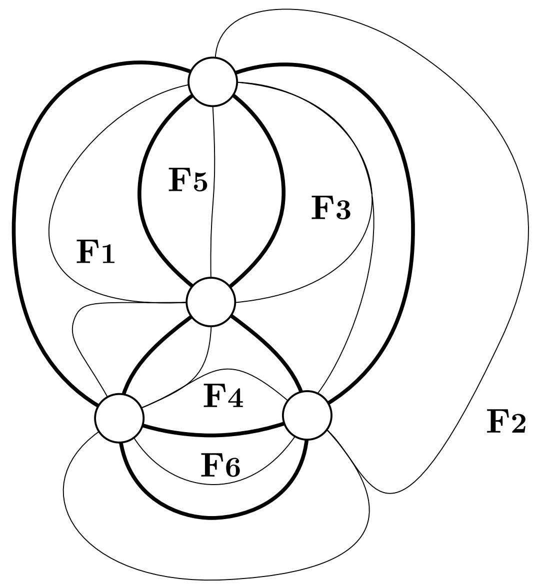


Figure 6.1 (f)

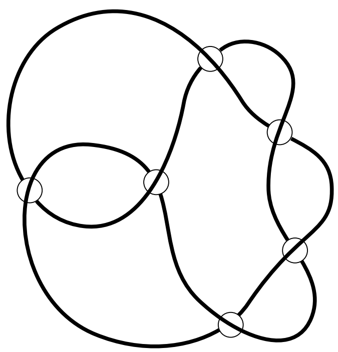
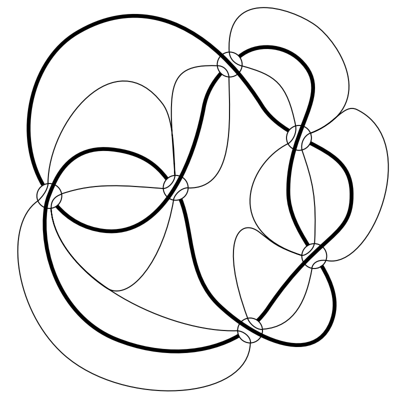
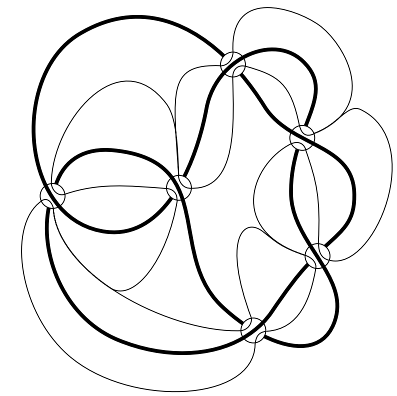


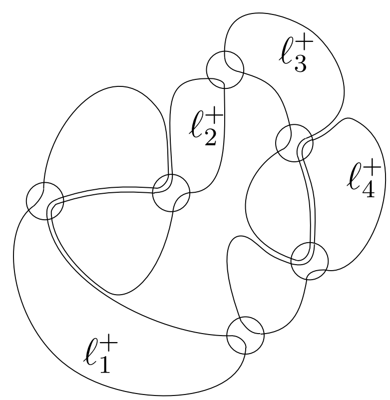
Figure 6.2 (a)



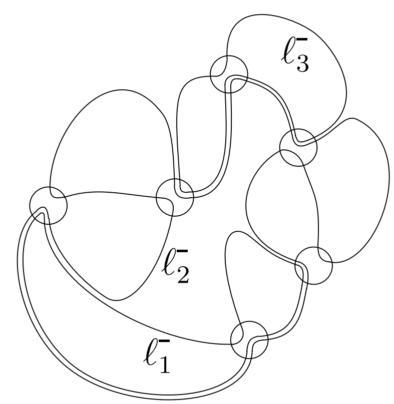
(b)



(c)



(d)



(e)

Figure 6.2

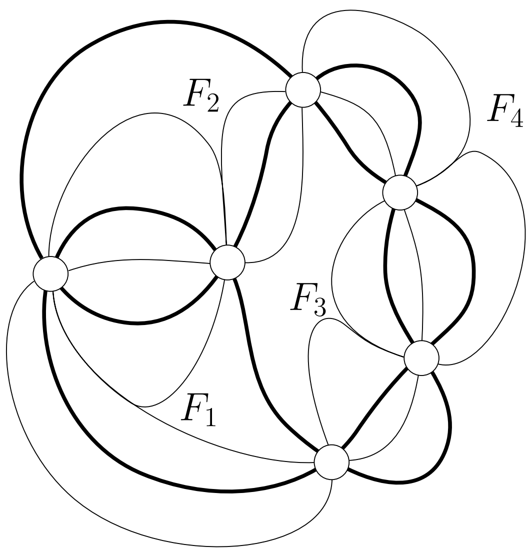
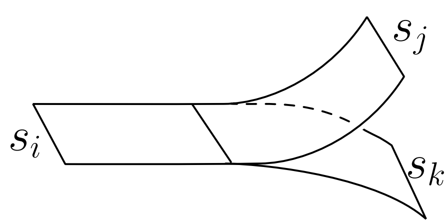


Figure 6.2 (f)



$$w_i = w_j + w_k + 1$$

Figure B1

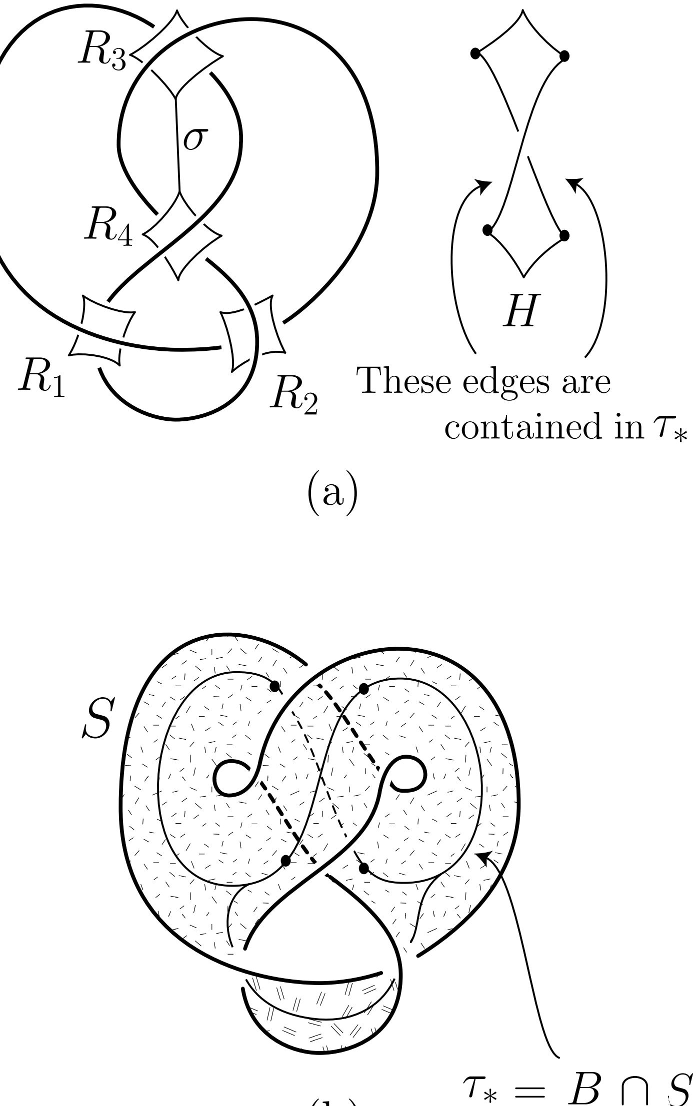


Figure D1

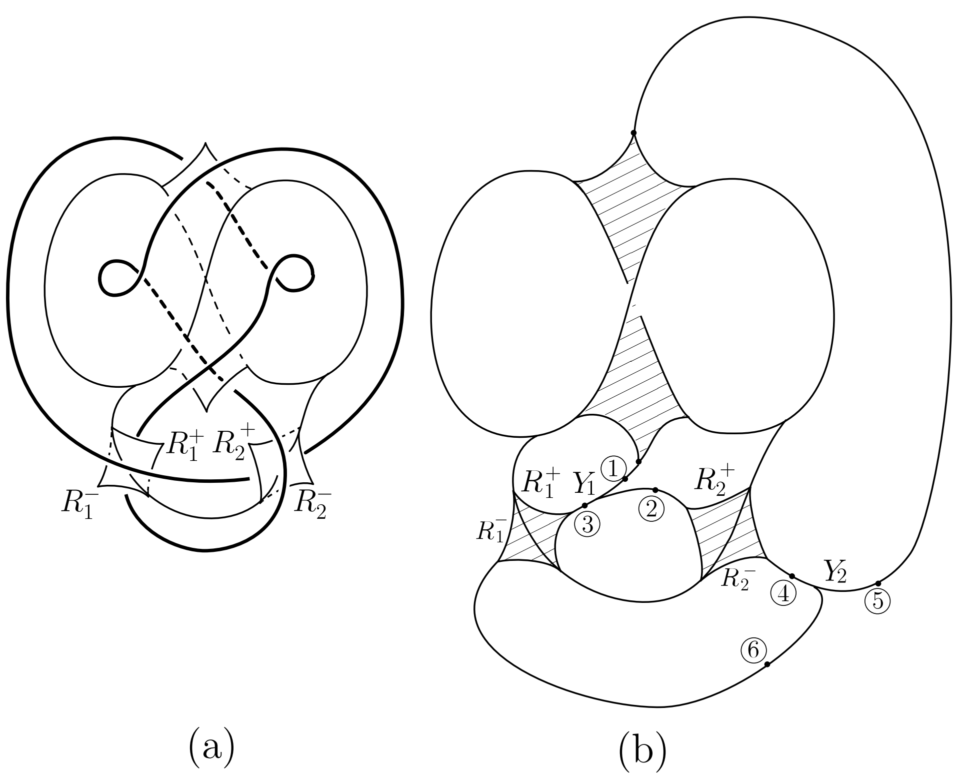


Figure D2

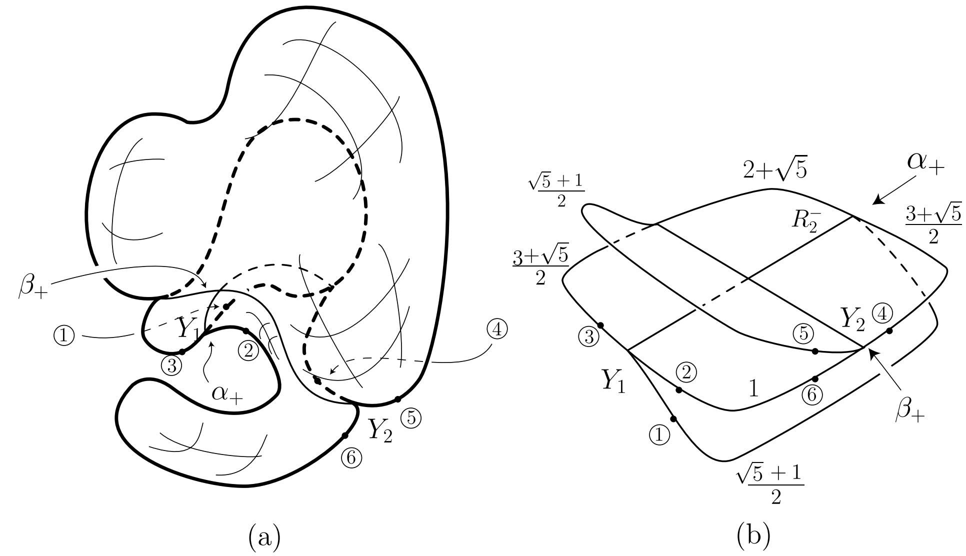
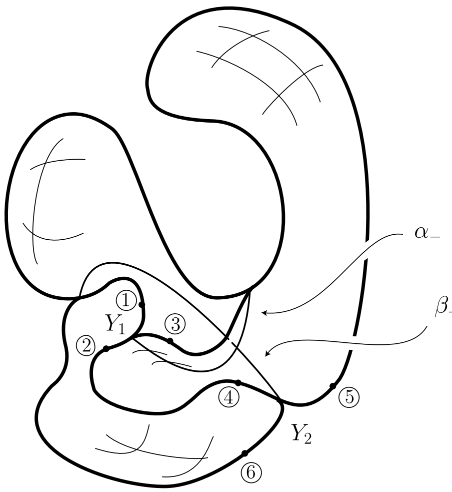
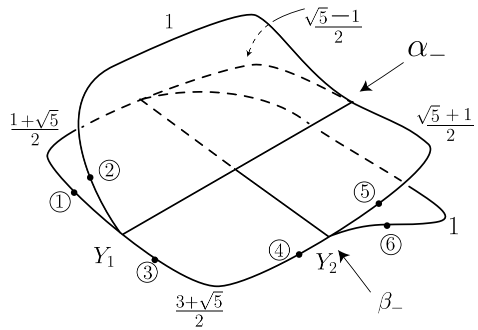


Figure D3



(a)



(b)

Figure D4

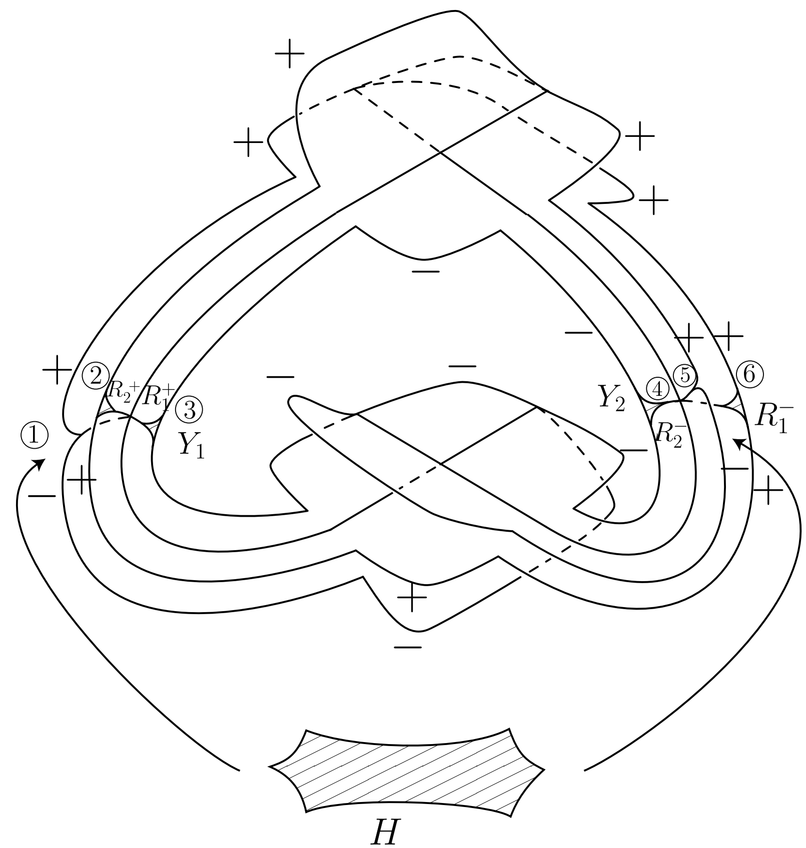


Figure D5

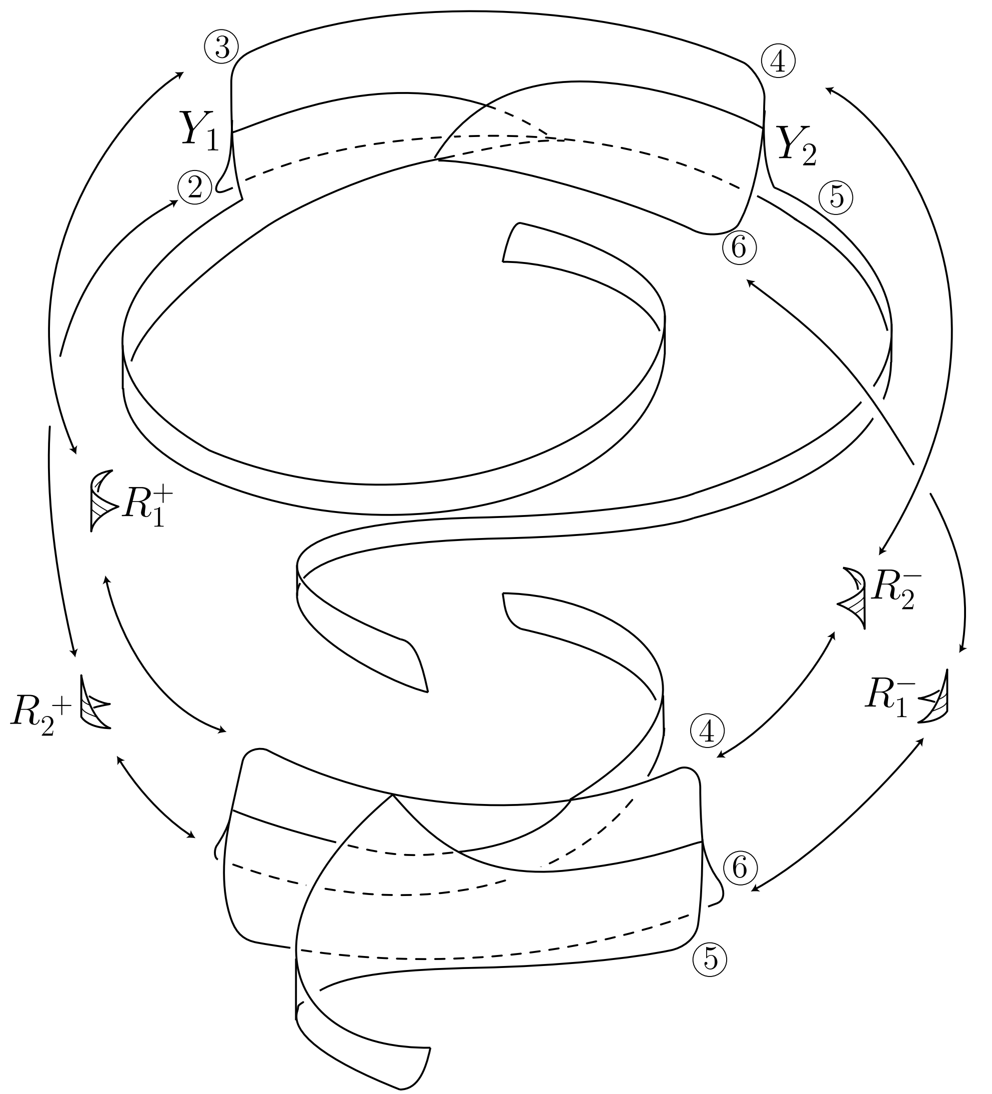


Figure D6 (a)

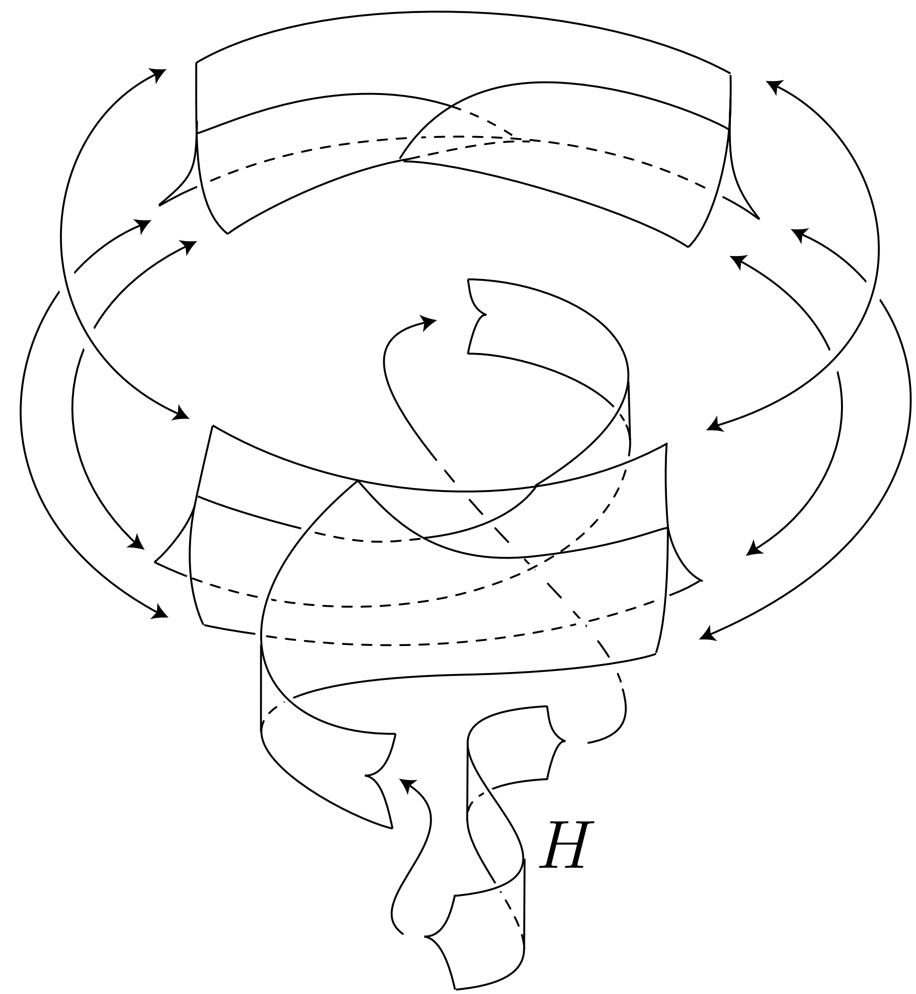
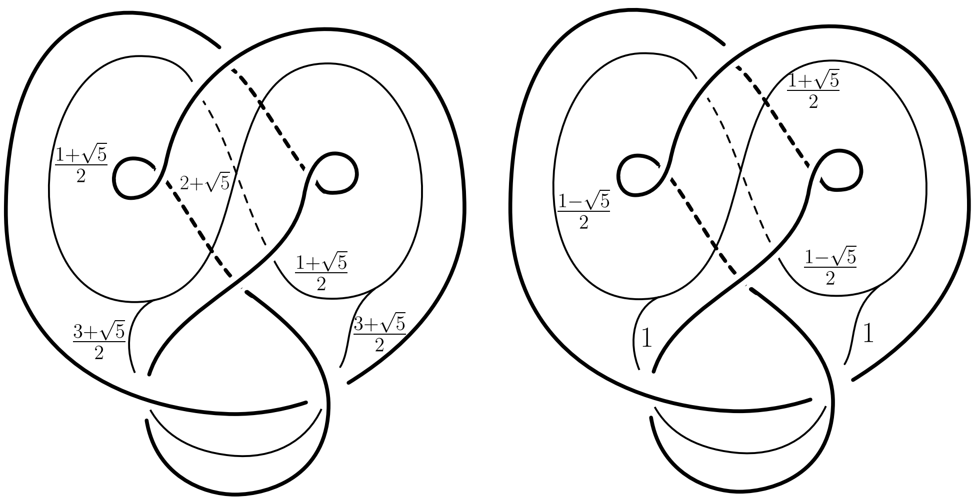
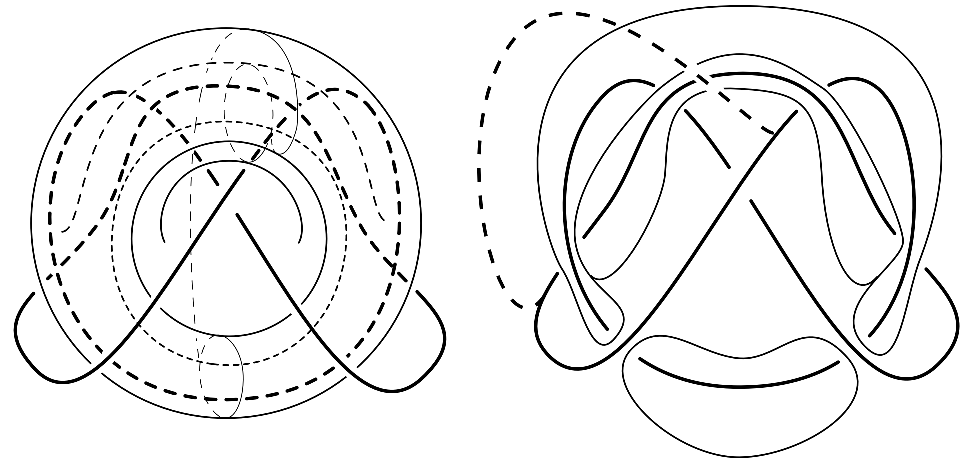


Figure D6 (b)



(a) (b)

Figure D7



(a) (b)

Figure E1