Essential Laminations in Seifert-fibered Spaces

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0. Introduction

Over the past few decades, the importance of the incompressible surface in the study of 3-manifold topology has become apparent. In fact, nearly all of the important outstanding conjectures in the field have been proved, for 3-manifolds containing incompressible surfaces (see, e.g., [20],[22]). Faced with such success, it becomes important to know just what 3-manifolds could contain an incompressible surface.

Historically, the first 3-manifolds (with infinite fundamental group) which were shown to contain no incompressible surfaces were a certain collection of Seifert-fibered spaces. Waldhausen [21], in the 1960's, showed that an incompressible surface in a Seifert-fibered space is isotopic to one which is either vertical or horizontal. This added structure puts a severe restriction on the existence of an incompressible surface, and led to the discovery of these 'small' Seifert-fibered spaces.

Now in recent years the essential lamination, a recently-defined hybrid of the incompressible surface and the codimension-one foliation without Reeb components, has begun to show similar power in tackling problems in 3-manifold topology (see [7]). It also has the added advantage of being (seemingly) far more widespread than either of its 'parents'; its more general nature makes it far easier to construct in a wide variety of 3-manifolds (see, e.g., [6]). In light of these facts, it would be interesting to know if there are any 3-manifolds which contain no essential laminations, and only natural to look in the same place that Waldhausen found his examples.

In this paper we carry out such a program. We show that an essential lamination in a Seifert-fibered space satisfies a structure theorem similar to the one given for surfaces by Waldhausen. Together with work of Eisenbud-Hirsch-Neumann on the existence of horizontal foliations, this structure theorem allows us to show that some of the 'small' Seifert-fibered spaces above cannot contain any essential laminations.

We also obtain, as a further application of the structure theorem, a result which states that any codimension-one foliation with no compact leaves in a 'small' Seifert-fibered space is isotopic to a horizontal foliation; this completes (in some sense) a group of results on isotoping foliations in Seifert-fibered spaces, which began with Thurston's thesis.

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1. The Main Results

For definitions and notations concerning essential laminations, see [7].

In this paper the word 'lamination' will mean a lamination which is carried by a branched surface; technically, therefore, a foliation $\mathcal{F}$, for example, is not a 'lamination'. One must first split $\mathcal{F}$ along a finite number of its leaves, as in [7]. Because we are largely interested in the existence of essential laminations, splitting will cause no difficulty; the splitting of an essential lamination is essential.

For definitions and basic concepts regarding Seifert-fibered spaces, see [8] or [16].

Generalizing [21], we say that a lamination $\mathcal{L} \subseteq$M is vertical, w.r.t. a Seifert-fibering p:M$\rightarrow$F if $p^{-1}(p(\mathcal{L}))=\mathcal{L}$, i.e., $\mathcal{L}$ contains every circle fiber of M that it meets; $\mathcal{L}$ is horizontal if it is transverse to the circle fibers of M at every point.

Now let M be a compact orientable Seifert-fibered space, with Seifert-fibering $\pi : M \rightarrow F$.

Theorem 1: Every essential lamination $\mathcal{L}$ in M contains a sublamination $\mathcal{L}_0$ which is isotopic to a vertical or horizontal lamination.

The proof of this theorem comprises the bulk of this paper.

Corollary 2: If a Seifert-fibered 3-manifold M contains an essential lamination, then it contains a horizontal or vertical one.

Therefore, if we wish to show that a Seifert-fibered space contains no essential laminations, it suffices to show that it contains no horizontal or vertical ones.

It is well-known that M contains a vertical essential surface unless either F=S^2 and M has $\leq$3 multiple fibers, or F=RP^2 and M has $\leq$1 multiple fiber. Of these cases the only one of interest is F=S^2 with 3 multiple fibers; in the remaining cases M is either reducible or has finite fundamental group [7], so cannot contain an essential lamination for well-understood reasons [7].

Proposition 3: There are no vertical essential laminations in a Seifert-fibered space M with base S^2 and 3 multiple fibers.

Proof: Suppose $\mathcal{L}$ is a vertical essential lamination. After splitting along some leaves of $\mathcal{L}$, we may assume that $\mathcal{L}$ misses the multiple fibers $\gamma_1, \gamma_2, \gamma_3$ of M, and so can be thought of as a (vertical) lamination in $M_0=M\setminus N(\gamma_1\cup\gamma_2\cup\gamma_3)=F\times S^1$, where F is a pair of pants $S^2\setminus 3D^2$. Because $\mathcal{L}$ is vertical (and M_0 has no multiple fibers), $\lambda=p(\mathcal{L})\subseteq F$ is a (1-dimensional) lamination in F. Further, because $\mathcal{L}$ is essential in M, it is easy to see that λ is incompressible in F; we can think of $F\subseteq M_0$ (by choosing a section of the (trivial) fibering of M_0) and λ=$\mathcal{L}\cap$F, and then any compressing or end-compressing disk for a leaf of λ will be a compressing or end-compressing disk for $\mathcal{L}$ in M, because $\mathcal{L}$ is vertical. But an easy Euler-characteristic calculation like those in [2] or [7], using an incompressible train track carrying λ, shows that any incompressible lamination in the interior of a pair of pants must contain a (∂-parallel) compact loop γ. But then $p^{-1}\gamma$=T is a vertical torus in $\mathcal{L}\subseteq$M, which bounds a solid torus (one of the N(γ_i)), and hence is compressible, a contradiction.

∎

Corollary 4: Every essential lamination $\mathcal{L}$ in a Seifert-fibered space M with base S^2 and 3 multiple fibers contains a horizontal sublamination.

Now it is easy to see that any horizontal lamination $\mathcal{L}$ can be completed to a transverse foliation of M; $\mathcal{L}$ cuts the circle fibers of M into arcs, so M split along $\mathcal{L}$, M|$\mathcal{L}$, is a collection of I-bundles, and these bundles can be foliated by surfaces transverse to the I-fibers, completing $\mathcal{L}$ to a foliation of M. Because the I-fibers are contained in the circle fibers of M, this foliation is everywhere transverse to the circle fibers of M.

In [4] and [11] such foliations were studied, and criteria based on the normal Seifert invariants of M were given for determining their existence. More precisely, suppose M is a Seifert-fibered space with normal Seifert invariant M= $\Sigma(0,0;k,a_1/b_1,a_2/b_2,a_3/b_3)$ and suppose either

(a) k$\neq$-1, -2, or

(b) k=-1, and (possibly after a permutation of the a_i/b_i) $a_i/b_i \geq a_i'/b_i' > 0$, for some rational numbers a_i'/b_i' satisfying

$$a_1'/b_1' = 1 - (a_2'/b_2' + a_3'/(b_2'(b_3'-1)))$$

or

(c) k=-2; then after replacing M= $\Sigma(0,0;-2,a_1/b_1,a_2/b_2,a_3/b_3)$ with M= $\Sigma(0,0;-1,(b_1-a_1)/b_1,(b_2-a_2)/b_2,(b_3-a_3)/b_3)$ (by reversing the orientation of M), apply the criterion (b).

Then M does not admit a transverse foliation.

In particular, M contains no essential laminations. Since it is well known that Seifert-fibered spaces M as above with $1/b_1+1/b_2+1/b_3<1$ have universal cover R^3 (see [16]), we have the following corollary.

Corollary 5: There exist Seifert-fibered spaces M with $\widetilde{M}=R^3$ which contain no essential laminations.

We now turn our attention to foliations without Reeb components of a Seifert-fibered space M.

Proposition 6: If an essential lamination $\mathcal{L}$ with no compact leaves in a closed, orientable Seifert-fibered space M contains a horizontal sublamination $\mathcal{L}_0$, then $\mathcal{L}$ is isotopic to a horizontal lamination.

Proof: Since $\mathcal{L}_0$ is horizontal, $M|\mathcal{L}_0$ is a collection of I-bundles foliated by subarcs of the circle fibers of M. Let N be a component of $M|\mathcal{L}_0$, an I-bundle over some non-compact surface E, $\pi: N\to E$, and consider $\mathcal{L}_1=\mathcal{L}\cap N\subseteq N$. Every leaf L of $\mathcal{L}_1$ is π_1-injective in N (since the composition $\pi_1(L)\to\pi_1(N)\to\pi_1(M)$ is injective).

Now let $\{C_i\}$ be an <u>exhaustion</u> of E by compact, connected subsurfaces, i.e., $\cup C_i=E$, and let $E_i=E\setminus \mathrm{int}(C_i)$. Because the leaves of $\mathcal{L}_1$ limit on leaves of $\mathcal{L}_0$ (in fact their limit set is contained in $\mathcal{L}_0$), which are horizontal, one can then see that for some i, every leaf of $\mathcal{L}_1$ is horizontal over E_i. So to show $\mathcal{L}$ can be made horizontal, it suffices to show that $\mathcal{L}\cap\pi^{-1}(C_i)$ can be isotoped to be horizontal in $N_i=\pi^{-1}(C_i)$, rel $\pi^{-1}\partial(C_i)=A$. Note that N_i is a compact handlebody.

We proceed by induction on the genus of N_i (see Figure 1). If genus=0, then C_i is a disk, and $N_i = C_i \times I$, with $C_i \times \partial I \subseteq \mathcal{L}_0$, and $\mathcal{L}_1$ meeting $\partial C_i \times I$ horizontally. Therefore $\mathcal{L}_1 \cap N_i$ is a collection of taut disks, which can be pulled horizontal.

Figure 1: horizontal laminations

If genus> 0, then choose an essential arc α in C_i and look at the disk $\Delta = \pi^{-1}\alpha$. $\partial\Delta$ can be separated into four arcs, two contained in $\mathcal{L}_0$ and two transverse to $\mathcal{L}_1$. By an isotopy of $\mathcal{L}_1$ we can remove any trivial loops of intersection $\mathcal{L}_1 \cap \Delta$; then $\mathcal{L}_1$ meets Δ in compact arcs. None of these arcs can have both endpoints in the same arc of $\partial\Delta$; the disk it cuts off together with a (vertical) half-infinite rectangle going off to infinity in N would give an end-compressing disk for $\mathcal{L}$.

So all of the arcs run from one side of Δ to the other; in particular, these arc can be pulled taut w.r.t. the I-fibering of Δ from N. If we then split open N_i along Δ, we get an I-bundle of smaller genus, with $\mathcal{L}_1$ meeting the ∂I-bundle horizontally, and with horizontal complement in N. By induction, therefore, we can isotope $\mathcal{L}_1$ (rel $\mathcal{L}_0$) to be horizontal in N. Doing this for all of the components of $M|\mathcal{L}_0$, we can isotope $\mathcal{L}$ to be horizontal in M. ∎

Corollary 7: Every essential foliation with no compact leaves $\mathcal{F}$ in a Seifert-fibered space M with base S^2 and 3 multiple fibers is ($C^{(0)}$-) isotopic to a transverse foliation.

Proof: We can split $\mathcal{F}$ along a finite collection of leaves to give an essential lamination $\mathcal{L}$ carried by a branched surface. By the corollary above, $\mathcal{L}$ contains a horizontal sublamination. By the proposition (since $\mathcal{L}$ has no compact leaves) $\mathcal{L}$ itself is a horizontal lamination. The I-bundles $M|\mathcal{L}$ then are fibered by arcs in the circle fibers; crushing each fiber to a point retrieves $\mathcal{F}$ in M, and it is now transverse to the fibers of M. ∎

This result can be thought of as an extension and completion (in the C^0-case) of results of Thurston [19] and Levitt [12], Eisenbud-Hirsch-Neumann [4], and Matsumoto [13]. Taken together these papers show that a C^2-foliation with no compact leaves, in any (closed) Seifert-fibered space other than the ones in the corollary, can be C^2-isotoped to a transverse one. The corollary says that a C^2-foliation in M with base S^2 and 3 multiple fibers can be C^0-isotoped to a transverse one; it leaves open the question of whether such a foliation can be C^2-isotoped (the argument above cannot be adapted; at the very beginning, the splitting of the foliation to obtain a branched surface destroys the transverse C^2-structure).

It is worth noting that an extension in the other direction is not possible; there exist C^0-foliations of Seifert-fibered spaces, with no compact leaves, which contain vertical sublaminations. Examples are easily constructed from vertical essential laminations in $F \times S^1$, where F is a compact surface of genus greater than or equal to 2.

2. Proof of the Theorem: Preliminaries

Every orientable Seifert-fibered 3-manifold M is the union of a (finite) collection of solid tori (with disjoint interiors) which meet along their boundaries. This view can be obtained from the standard one. Consider the base surface F of the Seifert-fibering; it is a compact surface. Choose a triangulation τ of F, in general position with respect to the collection of multiple points of the fibering, so that each 2-simplex contains at most one multiple point. Then every 2-simplex Δ_i^2 has inverse image $\pi^{-1}(\Delta_i^2) = M_i$ a solid torus (it is an (orientable) Seifert-fibered space with base D^2 and at most one multiple fiber), and these solid tori meet along the inverse images of the 1-simplices of τ, which meet each solid torus in its boundary.

The inverse images of the points of $\tau^{(0)}$ =the 0-skeleton of τ form a finite collection S of regular fibers of M in the boundary of the solid tori (they in fact constitute the points where three or more of the solid tori meet). These fibers will be of central importance to us; we will call them the <u>sentinel</u> <u>fibers</u> of M.

If the lamination $\mathcal{L}$ is carried by a branched surface B, then possibly after splitting along a finite number of leaves, we may assume that $\partial_h N(B) \subseteq \mathcal{L}$. Then N(B) split open along $\mathcal{L}$, denoted $N(B)|\mathcal{L}$, is a collection of I-bundles over compact and non-compact surfaces (possibly with boundary). If we split B along the bundles over compact surfaces (i.e., remove their interiors from N(B)), we obtain a (possibly new) branched surface B, carrying $\mathcal{L}$, which now has no such bundles in $N(B)|\mathcal{L}$. Such a branched surface will be called a branched surface having no compact bundles w.r.t. $\mathcal{L}$. Every lamination (up to splitting) is carried by such a branched surface, except when it has a compact isolated leaf.

For a lamination $\mathcal{L} \subseteq M$ carried by a branched surface B having no compact bundles w.r.t. $\mathcal{L}$, and γ a loop transverse to $\mathcal{L}$ (i.e., to B), we can define a number ϵ, called a <u>monogon</u> <u>number</u> for $\mathcal{L}$ w.r.t. γ, in terms of the branched surface B, as follows:

N(B) meets γ in a collection of vertical fibers, and $\mathcal{L} \cap \gamma$ is contained in these subarcs of γ; we let $\epsilon = 1/2$ of the smallest distance (along γ) from one of these subarcs to another. It then follows that any two points of $\mathcal{L} \cap \gamma$ which are within ϵ of one another are contained in the same vertical fiber of N(B).

We also need to know something about how $\mathcal{L}$ meets typical surfaces S in the M_i (that it meets transversely), i.e., (meridian) disks, annuli=$\pi^{-1}(e_j^1) \subseteq \partial M_i$, and tori=$\partial M_i$. Because $\mathcal{L}$ is essential, this is easy to categorize.

λ=$\mathcal{L}\cap$S is a 1-dimensional lamination in the surface S. There can be no holonomy around a loop γ of λ which is trivial in S (see [15] for the notion of holonomy), because γ bounds a disk in $\mathcal{L}$ and there can be no holonomy around the boundary of a disk. It follows by Reeb stability [17] that the collection λ_0 of trivial loops and ∂-parallel arcs in $\lambda \subseteq$S form a sublamination open and closed in λ.

$\lambda \setminus \lambda_0 \subseteq$S can have no monogons; because $\mathcal{L}$ is transverse to S they would give an end-compressing disk for $\mathcal{L}$. An Euler-characteristic argument like that in [2] implies (since χ(S)$\geq$0) that $\lambda \setminus \lambda_0$ can be completed to a foliation of S.

If S=torus, facts from dynamical systems about foliations of the torus (see, e.g., [9]) imply that there can be only 3 kinds of behavior in $\lambda \setminus \lambda_0$: either

(a) $\lambda \setminus \lambda_0$ contains no compact leaves; it then contains an irrational (measured) sublamination, and all other leaves are parallel to this sublamination. In particular, $\lambda \setminus \lambda_0$ can be isotoped to be transverse to the leaves of any foliation of S by compact loops (since this is true of the completed foliation of S); or

(b) $\lambda \setminus \lambda_0$ contains compact leaves. The collection of compact leaves then forms a (closed) sublamination of $\lambda \setminus \lambda_0$, and all other leaves lie in the annular regions between the compact leaves, and are of either (1)'Reeb' type or (2) 'Kronecker' type (see Figures 2ab).

Figure 2: laminations in the 2-torus

If S=disk, then $\lambda \setminus \lambda_0 = \emptyset$ (you can't foliate a disk), so all of the leaves of $\mathcal{L}\cap$S are trivial loops and ∂-parallel arcs. If S=annulus, then by doubling S and

$\lambda \setminus \lambda_0 \subseteq$S, and applying the above, we can conclude that either $\lambda \setminus \lambda_0$ consists of parallel essential compact arcs, or $\lambda \setminus \lambda_0$ contains essential (vertical) loops, with (possibly half-) Reeb or Kronecker leaves in between them.

a. Recognizing good laminations in a solid torus.

Let $\mathcal{L}$ be an essential lamination in the 3-manifold M, and let M_0 be a solid torus in M. By a small isotopy of $\mathcal{L}$ we can arrange that $\mathcal{L}$ is transverse to M_0 (this amounts to making $\mathcal{L}$ transverse to ∂M_0). Then $\mathcal{L} \cap M_0 = \mathcal{L}_0$ is a lamination in M_0. This lamination is almost certain not to have π_1-injective leaves. However, this lack of π_1-injectivity, basically, lives in the boundary $\partial\mathcal{L}_0 = \mathcal{L} \cap \partial M_0$, as the following lemma shows:

Lemma 2.1: Let $\mathcal{L}$, M_0, and $\mathcal{L}_0$ be as above, with $M \setminus \text{int}(M_0)$ irreducible. If every embedded loop γ_0 in $\partial\mathcal{L}_0$ which is null-homotopic in M_0 bounds a disk in $\mathcal{L}_0$, then every leaf of $\mathcal{L}_0$ is π_1-injective in M_0.

Proof: Look at the collection τ of loops in $\partial\mathcal{L}_0 = \mathcal{L}_0 \cap \partial M_0$ which are trivial in ∂M_0. These loops, by hypothesis are trivial in the leaves of $\mathcal{L}_0$ which contain them, and therefore bound (a collection $\mathcal{T}$ of) boundary-parallel disk leaves in $\mathcal{L}_0$. By a Reeb Stability argument, this collection $\mathcal{T}$ forms an open and closed set in $\mathcal{L}_0$, and so is a sublamination of $\mathcal{L}_0$, and so τ is a sublamination of $\partial\mathcal{L}_0$. τ therefore consists of a finite number of parallel families of trivial loops in ∂M_0, bounding parallel families of ∂-parallel disks in M_0. We can then by an isotopy of $\mathcal{L}$ (choosing an outermost family of disks (meaning an innermost family of loops) and working in) remove these families of disks from $\mathcal{L}_0$. Since $\mathcal{T}$ is closed in $\mathcal{L}_0$, nothing else is changed, so after the isotopy $\mathcal{L}_0$ has been altered to $\mathcal{L}_0 \setminus \mathcal{T}$, i.e., $\partial\mathcal{L}_0$ no longer contains any loops trivial in ∂M_0. We will prove the lemma for this altered lamination.

Let γ be a (singular) loop in a leaf L_0 of $\mathcal{L}_0$, which is null-homotopic in M_0. Then γ is also null-homotopic in M. Since L_0 is contained in a leaf L of $\mathcal{L}$, and this leaf is π_1-injective in M, it follows then that γ is null-homotopic in L. Let $H : D^2 \to L$, be a null-homotopy, and make it transverse to ∂M_0. Then , $= H^{-1}(\partial M_0)$ is a (finite) collection of circles in a disk D^2. Consider a circle γ_0 of , innermost in D^2, and consider the leaf l_0 of the lamination $\lambda_0 = \mathcal{L}_0 \cap \partial M_0$ which H maps it into. This leaf is homeomorphic to either S^1 or $\mathbf{R}$. If it is homeomorphic to $\mathbf{R}$, then γ_0 is null-homotopic in l_0, and so by redefining H on the disk Δ_0 of D^2 cut off by γ_0 so that it maps into l_0 (and then homotoping H off of l_0 slightly), we get a new null-homotopy for γ_0 with fewer circles of intersection in , . If l_0 is a circle, then one of two things will be true. In the most usual case γ_0 again maps into l_0 null-homotopically, in which case we proceed as before, finishing the proof by induction. When γ_0 is essential in l_0, we must use a different argument which avoids the induction.

Because γ_0 is innermost, it bounds a disk Δ_0 in D^2 which misses , , so the image of Δ_0 under H misses ∂M_0, and hence maps into M_0 or $M_1 = M \setminus \mathrm{int}(M_0)$. So some non-trivial power of l_0 is null-homotopic in M_0 or M_1.

If γ_0 is null-homotopic in M_1, this means that the torus ∂M_1 is <u>compressible</u> in M_1. Because M_1 is irreducible by hypothesis, it follows that M_1 is in fact a solid torus. This implies that our original 3-manifold M is a union of two solid tori glued along their boundary, and hence is a lens space. But this is impossible, since a lens space cannot contain an essential lamination (it does not have universal cover $\mathbf{R}^3$ [7]).

If γ_0 is null-homotopic in M_0, then because $\pi_1(M_0)$ is torsion-free (it's $\mathbf{Z}$), l_0 is also null-homotopic in M_0, and therefore bounds a disk leaf of $\mathcal{L}_0$, by hypothesis. By our additional hypothesis, this disk is not ∂-parallel in M_0, so it must be essential in M_0; in particular, $\mathcal{L}_0$ contains a meridian disk leaf. Now consider the collection μ of meridian loops of $\partial\mathcal{L}_0$. By hypothesis, these loops bound a collection $\mathcal{M}$ of meridian disk leaves of $\mathcal{L}_0$. Again, Reeb Stability implies that this collection $\mathcal{M}$ is closed in $\mathcal{L}_0$, so μ is closed in $\partial\mathcal{L}_0$. But then the leaves of $\partial\mathcal{L}_0$ not in μ live in the annular regions between loops of μ; they cannot be compact (they would then be trivial or meridional), but they cannot be non-compact, because they would have to limit on μ, giving non-trivial holonomy around a loop which bounds a disk. Therefore, $\mathcal{M} = \mathcal{L}_0$, so every leaf of $\mathcal{L}_0$ is a meridional disk, which obviously π_1-injects. ∎

b. Making intersections taut: solid tori.

Because a Seifert-fibered space can be thought of as a union of solid tori, which meet along their boundaries, it will also be useful to have a general procedure to isotope an essential lamination $\mathcal{L}$ so that it meets a vertical solid torus M_0 in a Seifert-fibered M in a lamination, $\mathcal{L} \cap M_0 = \mathcal{L}_0$, which has π_1-injective leaves. We will show later that such a lamination $\mathcal{L}_0$ in fact has a rather simple structure; this result will then be exploited to give our structure theorem for essential laminations in Seifert-fibered spaces.

Now there is in fact a very easy way to do this: just think of a solid torus M_0 as a regular neighborhood of its core circle γ_0, make γ_0 transverse to a branched surface carrying $\mathcal{L}$, and then $\mathcal{L} \cap M_0$ will be a collection of meridian disks in M_0, which certainly has π_1-injective leaves.

Unfortunately, this is a far too destructive process for our uses; it loses alot of the information that we will be gathering in the proof of our theorem. Instead we will construct an isotopy which is much more 'conservative' (and which, incidentally, allows much more interesting laminations $\mathcal{L} \cap M_0$ to be created).

We have seen already that in order to make a lamination meet a (nice) solid torus M_0 in a π_1-injective lamination $\mathcal{L}_0$=$\mathcal{L}\cap M_0$, we need only arrange that any loop of $\partial\mathcal{L}_0$ which is null homotopic in M_0 bounds a disk in $\mathcal{L}_0$. What we will now do is to describe an isotopy process which, given an essential lamination, will arrange exactly that.

First we deal with trivial loops of $\lambda_0 = \partial\mathcal{L}_0$. If $\partial\mathcal{L}_0$ contains loops which are trivial in ∂M_0, the collection C of such loops in ∂M_0 is open and closed in $\partial\mathcal{L}_0$, and (by transversality) consists of a finite number of families of parallel loops in $\partial\mathcal{L}_0$.

Now take an *outermost* loop γ of an *innermost* family of trivial loops. γ bounds a disk D in ∂M_0, and a disk D_0 in the leaf of $\mathcal{L}$ containing it, and they are isotopic, rel γ (because M is irreducible). An (ambient) isotopy of $\mathcal{L}$ taking D_0 to D and a bit beyond has the effect of removing the family of loops containing γ from λ_0 (and possibly more). To be more exact, such an isotopy must be done in stages, since it is not immediate that $D_0 \cap D = \gamma$; it could consist of (a finite number of) loops in D_0 (one then argues from innermost out). Then by induction on the number of parallel families in λ_0, we can assume that $\mathcal{L}\cap F$ contains no trivial loops.

Now if $\partial\mathcal{L}_0$ still contains loops which are null-homotopic in M_0, then these loops must be meridional, i.e., bound disks in M_0 but not in ∂M_0. What we first must establish is that at least one of these loops in $\partial\mathcal{L}_0$ bounds a disk leaf of $\mathcal{L}_0$.

Choose a meridional loop γ of $\partial\mathcal{L}_0$. Because $\mathcal{L}$ is essential, this (embedded) loop bounds a disk D in $\mathcal{L}$. Consider the intersection $D \cap \partial M_0 \subseteq \partial\mathcal{L}_0$; this intersection consists of (a finite number of) closed loops. Choose an innermost such loop γ_0 in D, bounding a disk Δ in D (possibly $\gamma_0 = \gamma$).

Claim: Δ is contained in M_0.

If not, then $\Delta \subseteq M \backslash \text{int}(M_0)$ (because γ_0 is innermost). γ_0 cannot be trivial in ∂M_0 (there are no trivial loops in λ_0, so it is essential in $\partial M_0 = \partial(M \backslash \text{int}(M_0))$. So Δ represents a compressing disk for $\partial(M \backslash \text{int}(M_0))$. Therefore, $M \backslash \text{int}(M_0)$ is a solid torus, making M a lens space (the union of two solid tori), a contradiction (a lens space doesn't have universal cover R^3, and M does [7]).

Therefore there is a disk Δ in $\mathcal{L}_0 \subseteq M_0$ with boundary a loop $\gamma_0 \subseteq \partial\mathcal{L}_0 \subseteq \partial M_0$. Consider now the collection $\mathcal{M}$ of meridian disk leaves of $\mathcal{L}_0$. Reeb Stability implies that this collection is open and closed in M_0, as before. Moreover, because $\mathcal{M} \neq \emptyset$, the lamination $\mathcal{L}_1 = \mathcal{L}_0 \backslash \mathcal{M}$ must have $\partial\mathcal{L}_1$ consisting of meridional loops; it cannot contain any trivial loops, by construction, and any non-compact leaf of $\partial\mathcal{L}_1$ would have to limit on a meridional loop, implying non-trivial holonomy around a loop null-homotopic in a leaf of $\mathcal{L}$, a contradiction.

Every leaf of $\mathcal{L}_1$ has more than one boundary component; if a leaf had only one and were compact, then it would contain a non-separating loop contained in the ball $M_0 \backslash \Delta$, implying the leaf of $\mathcal{L}$ containing it was not π_1-injective in M. If the leaf were non-compact, then the limit set of an end (see [14] for a definition) would be a lamination which did not meet ∂M_0; it would then be contained in the interior of a ball, implying the existence of an essential lamination in a sphere, which is impossible.

This implies that, although $M_0 | \mathcal{M}$ is a possibly infinite collection of balls, only finitely many of them can contain any leaves of $\mathcal{L}_1$. To see this, look at a loop γ having intersection number 1 with each loop of the meridional lamination $\partial\mathcal{L}_0$. If there are an infinite number of regions containing leaves of $\mathcal{L}_1$, then there are an infinite number of (distinct) arcs of $\gamma | \mathcal{M}$ meeting these leaves. Such a collection of arcs must have their lengths tending to 0. If we look at the top endpoints (in some orientation of γ) of these arcs, we have an infinite sequence of (distinct) points in $\partial\mathcal{M}$, which (because $\mathcal{M}$ is closed) must limit on some point of $\gamma \cap \mathcal{M}$. This point therefore lies in a meridional disk leaf D of $\mathcal{L}_0$; therefore by Reeb Stability, all nearby leaves are also meridian disks. But the top endpoints of the arcs are limiting on this leaf, and the lengths of the arcs are tending to zero (so the bottom endpoints are limiting on D, too), implying that these non-disk leaves pass arbitrarily close to D, a contradiction.

Now look at a component N of $M_0|\mathcal{M}$, and the leaves of $\mathcal{L}_1$ contained in it. N is a ball with two leaves of $\mathcal{M}$ in its boundary.

Every loop of $\partial\mathcal{L}_1 \cap N = \lambda$ bounds a disk D in the leaf of $\mathcal{L}$ containing it; thinking of $\mathcal{L} \subseteq N(B)$, the set of these disks which are parallel to D in N(B) have boundaries forming an open and closed set in λ. Consequently, they fall into finitely-many parallel families (in N(B)). For (choosing an arc β running from the top to the bottom of the ball) every point of $\lambda \cap \beta$ has an open neighborhood in β whose points are in loops bounding parallel disks in N(B); because $\lambda \cap \beta$ is compact in β, there is a finite subcover, giving the finite number of families.

Therefore, the loops of $\partial\mathcal{L}_0 \setminus \partial\mathcal{M}$ fall into a finite number of such parallel families.

It is possible to see a finite sequence of surgeries of $\mathcal{L}$ in M_0 which makes every loop in ∂M_0 bound a disk in M_0 (see Figure 3). These surgeries represent our 'template'; what we wish to do now is use this surgery picture to find an isotopy of $\mathcal{L}$ which will do the same thing.

Figure 3: surgery in the solid torus

We have a finite number $\lambda_1, \ldots, \lambda_n$ of families of loops in $\partial\mathcal{L}_0 \backslash \partial\mathcal{M}$ which bound a collection $\mathcal{D}_i$ of disks in $\mathcal{L}$ parallel in N(B). Think of doing these surgeries family by family. Choose a collection $\mathcal{D}_i$; note that every disk in the collection meets λ_i only in its boundary (a disk cannot be parallel in N(B) to a proper subdisk of itself - it would imply that the disk met an I-fiber of N(B) infinitely often).

Therefore the disk in $\mathcal{D}_i$ together with one of the disks from the surgery form an embedded sphere in M (all of which are parallel to one another); because M is irreducible, they bound (nested) balls (see Figure 4). This ball, together with the ball that the two 'outermost' surgery disks bound, forms a ball which can be used to describe an isotopy taking the disks in $\mathcal{D}_i$ to the (other) collection of disks in M_0, making the collection of loops λ_i bound disks in M_0. This isotopy may have removed leaves of $\mathcal{M}$, as well as loops from some of the λ_i, but since it can be thought of as a replacement (surgering, and then throwing away the spheres created), it adds nothing to any intersection $\mathcal{L}$ has with any object outside of the

Figure 4: surgeries to isotopies

interior of M_0; in particular, it adds no new intersections with ∂M_0, and moves none of the disks which it didn't erase. By a finite application of this process, then, we can arrange that every loop in $\mathcal{L} \cap \partial\mathcal{L}$ bounds a disk leaf in M_0, completing our isotopy.

3. π_1-injective, end-incompressible laminations in a solid torus

We have seen how to isotope an essential lamination $\mathcal{L}$ to make it meet a solid torus in a π_1-injective lamination $\mathcal{L}_0$ with no ∂-parallel disk leaves. It is easy to see that $\mathcal{L}_0$ is end-injective (this is in fact true for any transverse intersection of an essential lamination with a codimension-0 submanifold); any end-compressing disk for $\mathcal{L}_0$ is an end-compressing disk for $\mathcal{L}$. $\mathcal{L}_0$ is in general, however, not ∂-injective.

Such a lamination, however, still has a great deal of identifiable structure.

Theorem 3.1: A lamination $\mathcal{L}_0$ as above is either a collection of meridian disks, or there is a (model) Seifert-fibering of M_0 so that (after isotopy) $\mathcal{L}_0$ contains a vertical sublamination $\mathcal{L}_1$ (whose leaves are annuli, with possibly one Möbius band); all leaves of $\mathcal{L}_0 \backslash \mathcal{L}_1$ are non-compact, simply-connected, and horizontal.

The proof contains two essential ingredients; first one needs that the ∂-lamination $\partial\mathcal{L}_0$ contains compact loops (which determine the regular fiber of the Seifert-fibering), and then that every such compact loop is in the boundary of a compact leaf of $\mathcal{L}_0$. The union of these leaves is the vertical sublamination $\mathcal{L}_1$. First, though, we need a small catalogue of basic facts, so that we can more easily recognize when these two things are happening.

a. Some basic facts about laminations in a solid torus

Fact 1: $\mathcal{L}_0 \subseteq D^2 \times S^1$ must meet the boundary torus; $\partial\mathcal{L}_0 \neq \emptyset$.

This is true more generally; an essential lamination cannot live in the interior of a handlebody. To see this, take a meridian disk D (or, in the general case, a compressing disk for one of the handles), and make $\mathcal{L}_0$ transverse to it. If $\mathcal{L}_0 \cap D = \lambda$ contains any compact loops, we can isotope $\mathcal{L}_0$ to remove them. So we can assume λ contains no compact loops. λ is carried by the train track $\tau = B_0 \cap D$ (where $\mathcal{L}_0$ is carried by B_0), and contains only non-compact leaves; an Euler characteristic calculation implies that, if $\lambda \neq \emptyset$, λ will contain a monogon, so $\mathcal{L}_0$ does, which is essential since $\mathcal{L}_0$ is transverse to D. So $\lambda = \emptyset$, so $\mathcal{L}_0$ misses D, implying that $\mathcal{L}_0$ is contained in a ball, **B** (or, inductively, is contained in the interior of a handlebody of lower genus). It is π_1-injective there (same argument as before), and contains no spheres ($\mathcal{L}$ didn't), and so all of its leaves are planes. Capping this ball off with a ball, we get a lamination in S^3, which is essential (because monogons can be pushed off the capping ball), a contradiction.

Fact 2: Every leaf L of $\mathcal{L}_0$ meets T= ∂M_0.

Otherwise, the closure $\overline{L}$ of L would give a π_1-injective lamination missing T. Applying the argument above to this sublamination gives the same conclusion, unless $\overline{L} \cap D$ contains a monogon; but then Euler-χ arguments will find a monogon for $\mathcal{L}_0 \cap D$ inside that one, which gives an end-compressing disk for $\mathcal{L}_0$, because $\mathcal{L}_0$ is transverse to D.

Fact 3: If a leaf L of $\mathcal{L}_0$ has more than one compact ∂-component, then it is an annulus.

This is standard; the two loops γ_1, γ_2 are parallel, otherwise one of them is trivial (making L a boundary-parallel disk). We can assume that they are oriented coherently, so that they represent the same free homotopy class in the boundary torus. Draw an arc α in the leaf joining the two components; then $\gamma_1 * \alpha * \overline{\gamma_2} * \overline{\alpha}$ is (almost) an embedded loop in L null-homotopic in $D^2 \times S^1$, hence bounds a disk in L. It follows that L is a disk with two arcs in its boundary identified, i.e. an annulus.

Fact 4: An annulus A with ∂A vertical (in a model fibering of a solid torus) is vertical.

This is also standard; from the previous argument it is easy to see that A is ∂-parallel, and so isotopic to an (of necessity vertical) annulus in the boundary of the solid torus. Pushing it back into the solid torus slightly, we see that A is isotopic to a (properly embedded) vertical annulus.

Fact 5: A non-orientable surface L with π_1(L)=**Z** and a compact ∂-component γ is a Möbius band.

Proof: Let p:L_0 →L be the orientable double cover of L. γ is orientation-preserving in L, so $p^{-1}(\gamma) = \gamma_1 \cup \gamma_2$, disjoint simple loops mapping homeomorphically down to γ under p. Being simple loops, they do not represent a proper power in $\pi_1(L_0)$ =**Z** [2]. So both represent the <u>generator</u> (up to reorienting the curves), hence are freely-homotopic. By [5], they are then <u>isotopic</u>, and cobound an annulus A in L_0. Since γ_1 and γ_2 are ∂-components, this implies that L_0 itself is an annulus, hence compact.

So p(L_0)=L is compact; by the classification of surfaces, it is therefore a Möbius band.

Fact 6: A Möbius band L with ∂L vertical (in a model fibering of a solid torus) is vertical.

This follows from a result of [18], which says that one-sided incompressible surfaces in a solid torus with a single boundary curve are determined up to isotopy by the slope of that curve (π_1-injective surfaces are incompressible). With this result in hand it remains then only to show that ∂L represents 2x the generator of π_1(solid torus), because a vertical π_1-injective Möbius band with that boundary slope can easily be constructed.

But this in turn follows readily from some π_1 considerations; let M=solid torus, and consider M_0=M\int(N(L)). It π_1-injects into M (since L is π_1-injective), is irreducible (since M is) and has boundary =(∂M\$\partial$N(L))∪($\partial$N(L)∩int(M)= $A_1 \cup A_2$ =annulus∪annulus=torus. So M_0 is a solid torus, and M=$M_0 \cup_{A_2}$ N(L).

Claim: The core of A_2 represents a generator of $\pi_1(M_0)$. For if the map $\pi_1(A_2) \to \pi_1(M_0)$ sends 1 to n, then by Van Kampen's theorem π_1(M) is equal to $\pi_1(M_0) *_{\pi_1(A_2)} \pi_1(N(L))$, and since the core of A_2 represents 2 in π_1(N(L)), this implies that π_1(M)= $\mathbf{Z}$ =(a,b : $a^2 = b^n$) = G. But the subgroup generated by a^2 is normal (a^2 commutes with both a and b), and G modulo this subgroup is (a,b : $a^2 = 1, b^n = 1$) = $\mathbf{Z}_2 * \mathbf{Z}_n$. But every quotient group of π_1(M)=$\mathbf{Z}$ is cyclic, implying n=1.

In particular, M_0 deformation retracts to A_2, so M deformation retracts to the regular neighborhood N(L) of L. Since ∂L represents 2×generator in π_1(N(L)) (it's parallel to the core of A_2), it therefore represents 2×generator in π_1(M).

Note also that there cannot be two disjoint such Möbius bands in a solid torus M, because any other L′ would be contained in the solid torus complement M_0 of the other. The boundary of L′ is parallel to ∂L in M, but ∂L now generates the fundamental group of M_0 (this is easy to see when L is vertical), and so ∂L′ cannot in fact bound a Möbius band in M_0 (the generator can't be divisible by 2).

b. Proof of the theorem

Lemma 3.2: Any π_1-injective, end-incompressible lamination $\mathcal{L}_0$ in a solid torus M_0 contains a compact ∂-leaf.

Proof: Suppose not; we know from Fact 1 above that $\partial\mathcal{L}_0$ is non-empty. From the catalogue of ∂-laminations in section 2, $\partial\mathcal{L}_0$ contains an irrational lamination, and so can be isotoped so that it is everywhere transverse to the meridional foliation.

Pick a meridian disk D in M_0. By an isotopy of $\mathcal{L}_0$ (supported away from $\partial\mathcal{L}_0$) we can make $\mathcal{L}_0$ transverse to D. By the usual argument, $\mathcal{L}_0 \cap$D= λ_0 consists of circles and arcs, and by an isotopy of $\mathcal{L}_0$ we can remove the circles of intersection, using the π_1-injectivity of $\mathcal{L}_0$. Pick an outermost arc α of this intersection. It cuts D into two disks, one of which D_0 meets $\mathcal{L}_0$ only in an arc of its boundary. The other arc of ∂D_0 lies in $\partial M_i | \partial\mathcal{L}_0$, and and splits the component containing it into two half-infinite rectangles. Pick one rectangle R, then it is easy to see that $D_0 \cup$R is an end-compressing disk for $\mathcal{L}_0$, because $\mathcal{L}_0$ is transverse to ∂M_i, contradicting the end-incompressibility of $\mathcal{L}_0$. ∎

Proposition 3.3: Every compact loop in $\partial\mathcal{L}_0$ is contained in a compact leaf of $\mathcal{L}_0$.

Proof: We will proceed by exhaustion. For a different proof, arguing by contradiction, see [1].

If the loop is null-homotopic in M_0, then it bounds a meridional disk leaf of $\mathcal{L}_0$. Therefore, we can assume that it is not meridional. If any leaf L containing a compact ∂-loop is non-orientable, then by Fact 5, it is a Möbius band, hence compact. If we split $\mathcal{L}_0$ along L, and then split M_0 along L, we get a new lamination in a new solid torus, with all the same essentiality properties that the originals had. But this lamination now has no Möbius band leaves, by Fact 5 ($\mathcal{L}_0$ had at most one). In other words, after possibly splitting $\mathcal{L}_0$ and M_0 , we can assume that $\mathcal{L}_0$ contains no non-orientable leaf with a compact ∂-component.

Now let Δ be a meridonal disk of M_0, which we may assume meets $\partial\mathcal{L}_0$ transversely, and meets each compact loop of $\mathcal{L}_0$ tautly. By an isotopy of $\mathcal{L}_0$ supported away from ∂M_0, we may make $\mathcal{L}_0$ transverse to Δ, and by a further isotopy we can remove any loops of $\mathcal{L}_0 \cap \Delta = \lambda$. λ then consists of compact arcs, which fall into a finite number of parallel families.

It is easy to see by inspection that the collection of compact loops $\mathcal{C}$ of $\partial\mathcal{L}_0$ is closed in $\partial\mathcal{L}_0$. But also the collection $\mathcal{C}_0$ of loops in $\mathcal{C}$ which are in the boundary of a compact leaf of $\mathcal{L}_0$ is open and closed in $\mathcal{C}$; the leaf L must be an annulus, because $\pi_1(L)$ injects in $\pi_1(M_0)=Z$. Call the boundary components of L γ_1 and γ_2. It is easy to see that an arc α of λ emanating from γ_1 has its other endpoint in γ_2 (otherwise L contains an orientation-reversing loop), and L split along α is a disk, with boundary $\delta_1 \cup \alpha \cup \delta_2 \cup \bar{\alpha}$. This disk then lifts to a disk in any nearby leaf in the normal fence over L; in particular, its boundary lifts to a closed loop. This implies that if there is a compact loop γ lying close enough to γ_1 (say), then δ_1 lifts to a closed loop in the leaf containing γ, so α and $\bar{\alpha}$ are mapped onto one another, so δ_2 also lifts to a closed loop. Therefore the leaf of $\mathcal{L}_0$ containing γ has two compact ∂-components; by Fact 3, it is then an annulus, hence compact. This shows that the set of loops bounding compact leaves is open in $\mathcal{C}$. Showing $\mathcal{C}_0$ is closed is easier; the set of compact leaves of a lamination $\mathcal{L}_0$ is always closed [15, Lemma 1.2], so its set of ∂-components is also closed .

Suppose now that $\mathcal{C}\backslash\mathcal{C}_0$ is not empty. It then follows from the above that there is an arc of λ emanating from an element γ of $\mathcal{C}\backslash\mathcal{C}_0$ whose other endpoint is in a non-compact leaf. Because $\mathcal{C}\backslash\mathcal{C}_0$ is closed, we can find an outermost such arc α (i.e., one cutting off a subdisk Δ_0 of Δ which misses $\mathcal{C}\backslash\mathcal{L}_0 \backslash(\gamma\cap\alpha)$). γ is isolated in $\mathcal{C}$ on the Δ_0-side, because $\mathcal{C}_0$ is closed; and the arcs of λ joining the loops of $\mathcal{C}_0$ to one another one the Δ_0-side fall into a finite number of parallel families, so there are a finite number of innermost such arcs (i.e., closest to α), contained in a finite number of annulus leaves of $\mathcal{L}_0$. If we remove small neighborhoods of these annuli, we split M_0into a finite number of solid tori (with $\mathcal{L}_0$ meeting each solid torus in a π_1-injective and end-incompressible lamination), and in the component containing γ, (what is left of) Δ_0 no longer meets any other compact loops.

Figure 5: Finding the other compact loop

Now look at the arc α and the leaf of $\partial\mathcal{L}_0$ it joins to γ. We must have a situation like one of those pictured in Figure 5. If the other endpoint is in a Kronecker leaf (Figure 5a), or in the 'inner half' of a Reeb leaf (Figure 5b), then it is easy to find an end-compressing disk for $\mathcal{L}_0$, a contradiction. If it is on the outer half of a Reeb leaf ℓ_0, then we will iterate our chase, to find a contradiction.

Notice first that all of the arcs of λ in Δ_0 must be parallel, otherwise we can find an end-compressing disk (Figure 6a). If we follow ℓ_0 around, it will return to Δ_0 again at a point x_0 after travelling at net 0-times around $\partial \mathrm{M}_0$vertically, and there is an arc α_0 joining ℓ_0 to the outer half of some other Reeb leaf ℓ_1 (otherwise we can find an end-compressing disk (Figure 6b,c)). We can continue this construction, finding a sequence of arcs α_{i} of λ, which have a limit α_∞ in λ. But it is easy to see (by lifting the picture to the universal cover of M_0 (Figure 6d)) that the endpoints of α_∞ are in the same leaf of $\partial\mathcal{L}_0$, and split off an arc β which wraps around $\partial \mathrm{M}_0$ a net 0-times longitudinally. Therefore, $\alpha_\infty \cup \beta$ is a null-homotopic simple loop in a leaf L of $\mathcal{L}_0$, so bounds a disk in L. But it is easy to see that our construction (the $\alpha_{\mathrm{i}}\cup$ the arcs of the ℓ_{i}) string together to form a half-line spiralling in on $\alpha_\infty \cup \beta$, implying non-trivial holonomy around the boundary of a disk, which is impossible.

Figure 6: various cases

Consequently, $\mathcal{C}_0$=$\mathcal{C}$, i.e., every compact loop of $\partial\mathcal{L}_0$ is contained in a compact leaf of $\mathcal{L}_0$. ∎

To complete the theorem, consider our 'essential' lamination $\mathcal{L}_0$. By Lemma 3.2, $\partial\mathcal{L}_0$ contains a compact loop γ. Choose the Seifert-fibering of the solid torus

M_0 whose regular fiber in ∂M_0 is isotopic to γ. Since every compact loop of $\partial\mathcal{L}_0$ is parallel to γ, we can, after an isotopy of of $\mathcal{L}_0$ supported near ∂M_0, assume that every compact loop of $\partial\mathcal{L}_0$ is a fiber of M. Now by the proposition every leaf of $\mathcal{L}_0$ which contains a compact ∂-loop is compact. They have vertical boundaries, and so by the facts above, each can be isotoped to be vertical in M. They can in fact be so isotoped simultaneously; the leaves fall into a finite collection of parallel families, and each family can be isotoped in turn, from the innermost out; think of isotoping the innermost leaf of the family to the boundary and then back in slightly; this is an ambient isotopy which makes the entire family vertical. Subsequent isotopies will be supported away from the ones which have already been straightened. This isotopy gives the vertical sublamination $\mathcal{L}_1$ of the theorem.

Now consider the leaves of $\mathcal{L}_0$ which are not in $\mathcal{L}_1$. These leaves all have non-compact boundary (which we assume runs transverse to the foliation of ∂M_0 by fiber circles), and so limit on leaves of $\mathcal{L}_1$. From holonomy considerations, this limiting takes place in a very simple way; see Figure 7a.

Figure 7: making the other leaves horizontal

Thus in each component M_1 of $M_0|\mathcal{L}_1$, it is possible to arrange the leaves of $\mathcal{L}_0$, by an isotopy supported away from ∂M_0, to meet a saturated neighborhood of the boundary of the component as in Figure 7b. It is easy then to see that $\mathcal{M}_0 = \mathcal{L}_0 \cap (M_1 \backslash \mathrm{int}(N(\mathcal{L}_1))$ is π_1-injective in $M_2 = M_1 \backslash \mathrm{int}(N(\mathcal{L}_1))$ (we have just removed half-infinite rectangular 'tails' from the leaves of $\mathcal{L}_0$, and the solid torus

M_2 π_1-injects into M_0), and end-incompressible (a monogon for $\mathcal{M}_0$ is a monogon for $\mathcal{L}_0$, since $\mathcal{L}_0$ is transverse to ∂M_1). Also, its ∂-lamination is transverse to the circle fibering of ∂M_1 induced from M_0, so it has no trivial leaves. Consequently, by the proof above, it either consists of meridian disks, or it contains an annulus or Möbius band leaf L. If the latter occurs, then L has boundary transverse to the vertical fibering of ∂M_2 induced from M_0, and so meets every fiber of ∂M_2. In particular, since $\partial M_2 \cap \partial M_0 \neq \emptyset$, ∂L meets ∂M_0.

Now, there is an arc α in L which together with an arc δ in ∂M_2 bounds a disk D in M_2 (if L is an annulus this is because it is ∂-parallel; if L is a Möbius band, look at the boundary of a regular neighborhood of L; it is a ∂-parallel annulus, which supports such a disk, and then project back). By an isotopy of D (leaving α in L and δ in ∂M_2) we can arrange that δ is contained in an annulus A of $\partial M_2 \cap \partial M_0$, and so we can make it lie in a circle fiber of this annulus. We may also assume that D is transverse to $\mathcal{L}_0$, meeting it in a collection of compact arcs.

Now consider in what leaves of $\partial\mathcal{L}_0 \subseteq \partial M_0$ the endpoints of δ are lying in. None of the circle loops of $\partial\mathcal{L}_0$ meet A, so these points are contained in (distinct; these leaves run transverse to the circle fibering of ∂M_0) non-compact leaves of $\partial\mathcal{L}_0$. Therefore (see Figure 7c) δ together with a pair of half-infinite arcs in $\partial\mathcal{L}_0$ cut off a half-infinite rectangle in ∂M_0; this together with the disk D form a monogon for $\mathcal{L}_0$; embedded in this is a end-compressing disk for $M_0|\mathcal{L}_0$ (essential because its 'tail' is in ∂M_0, which is transverse to $\mathcal{L}_0$).

This gives us a contradiction, so $\mathcal{M}_0$ consists of meridian disks with boundary transverse to the circle fibering; an isotopy rel boundary makes this a collection of <u>horizontal</u> disks. Doing this for all of the components of $M_0 \setminus \mathcal{L}_1$ gives an isotopy of $\mathcal{L}_0$ which makes every leaf of $\mathcal{L}_0 \setminus \mathcal{L}_1$ horizontal, in our chosen Seifert-fibering of M_0.

Since the lamination in the saturated neighborhood is also clearly horizontal, this implies that the leaves of $\mathcal{L}_0 \backslash \mathcal{L}_1$ can be isotoped, rel $\mathcal{L}_1$, to be horizontal in M_0. By gluing back, we have then arranged that

(*) the leaves in the complement of the vertical sublamination of $\mathcal{L}_0$ found above can be isotoped (rel the vertical sublamination) so that they are horizontal.

Since these leaves are just disks with half-infinite rectangles glued to them, they are also simply-connected. This completes our proof.

4. A special case: $\partial M \neq \emptyset$ and $\mathcal{L} \cap \partial M = \emptyset$

In this section we give a proof of the theorem in the case stated in the title. In the next section we give the general proof; this preliminary result will need to be used in that proof.

In this case in fact only one of the stated conclusions can occur:

Theorem 4.1: If $\mathcal{L}$ is an essential lamination in the compact, orientable Seifert-fibered space M, with $\partial M \neq \emptyset$ and $\mathcal{L} \cap \partial M = \emptyset$, then up to isotopy, $\mathcal{L}$ contains a vertical sublamination.

The idea of the proof (as in the general case) is to split M up into a collection of solid tori M_i, and then isotope $\mathcal{L}$ so that it meets each solid torus in a π_1-injective lamination $\mathcal{L}_i \subseteq M_i$ with no ∂-parallel disk leaves. In each solid torus it is therefore is an 'essential' lamination, and so our structure theorem of the previous section tells us what each looks like.

The proof here involves a somewhat different decomposition of M into solid tori than the one described in section 2. The base of the Seifert-fibering is a compact surface with boundary. It is well-known that such a surface can be split along proper arcs to give a disk; splitting along additional arcs, as necessary, we can split the surface into a collection of disks, each containing at most one multiple point(=image of a multiple fiber) of the Seifert-fibering. Then as before, the inverse images of these disks are solid tori; the difference here is that each of the solid tori of the decomposition meets ∂M in one or more annuli, and $\mathcal{L}$ does not meet these annuli (because it misses the boundary). Let A= the union of the inverse images of the splitting arcs; it is a finite union of annuli. By an isotopy of $\mathcal{L}$ we can make $\mathcal{L}$ transverse to A, and by the usual methods, we can remove any trivial circles of intersection from $\mathcal{L} \cap$A=λ. λ is then incompressible in A, so any compact loop in λ is parallel to ∂A; by an isotopy of $\mathcal{L}$ we can make such loops vertical in A. Set $\mathcal{L}_i$=$\mathcal{L}\cap$$M_i$.

Each $\mathcal{L}_i$ is π_1-injective in M_i, by Lemma 2.1, sine M\int(M_i) is Seifert-fibered with non-empty boundary, hence is irreducible (see, for example, [Ha]), and $\partial\mathcal{L}_i$ contains no meridonal loops (they would have to meet $M_i \cap \partial M$).

Therefore each $\mathcal{L}_i$ is an 'essential' lamination in the solid torus M_i which contains it. Now if $\mathcal{L}\cap A=\emptyset$, then $\mathcal{L}_i\cap\partial M_i=\emptyset$ for all i. But then by Fact 1, $\mathcal{L}_i=\emptyset$ for all i, so $\mathcal{L}=\emptyset$. If $\mathcal{L}\cap A\neq\emptyset$, then for some i, $\partial\mathcal{L}_i$ contains vertical loops, and so some $\mathcal{L}_i$ contains a vertical sublamination. Now consider all of the vertical sublaminations in all of the $\mathcal{L}_i$. They each meet ∂M_i in the (entire) collection of vertical loops of $\partial\mathcal{L}_i$, and so they glue together across the A_j to give a lamination in M, which is the vertical sublamination of $\mathcal{L}$ required by the theorem.

5. Proof in the general case

For convenience we will assume that M is closed; $\partial M = \emptyset$. The proof in the bounded case is entirely similar, although some of the isotopies must be constructed slightly differently.

We think of M as a union of (embedded) solid tori $M_i = \pi^{-1}(\Delta_i^2)$, i=1,...,r which meet one another in the annuli A_j in their boundaries. We set S=$\pi^{-1}(F^{(0)})$, the collection of *sentinel fibers* of the decomposition of M into solid tori.

a. The isotopy process

The strategy of the proof is to set up an isotopy process, i.e., a sequence of isotopies I_j which will, one by one, isotope $\mathcal{L}$ to meet the i^{th} solid torus ($j \equiv i$(mod r)) only in horizontal disks, while at the same time controlling the intersection of $I_j(\mathcal{L})$ with the sentinel fibers S. What we will see is that if at any stage of the process we are unable to continue the isotopy process, we can use this information to find a vertical sublamination of $\mathcal{L}$ (after possibly splitting one of the leaves of $\mathcal{L}$). Otherwise, we are able to continue the isotopy process indefinitely, and then we will be able to see that (larger and larger pieces of) $\mathcal{L}$ begin to limit on (larger and larger pieces of) some lamination $\mathcal{L}_0$, which, by its construction, is horizontal; as it turns out, $\mathcal{L}_0$ is in fact isotopic to a sublamination of $\mathcal{L}$.

We have seen how to isotope a lamination so that it meets a (vertical) solid torus M_i in a lamination $\mathcal{L}_i$ with π_1-injective leaves ($M \setminus M_i$ is irreducible because it is Seifert-fibered with non-empty boundary (see [7])). Consider now how this isotopy affects $\mathcal{L} \cap S$, the intersection of $\mathcal{L}$ with the sentinel fibers S. This isotopy was achieved by doing surgery on $\mathcal{L}$ in the solid torus, and then throwing away any 2-spheres which are created. In terms of the sentinel fibers, this means that $\mathcal{L} \cap S$ (after surgery) is contained in $\mathcal{L} \cap S$ (from before the surgery). This is what we mean by controlling the isotopies. We will call an isotopy which has this control conservative.

Now after this (preliminary) isotopy, we have arranged that $\mathcal{L} \cap M_i = \mathcal{L}_i$ is π_1-injective in M_i. It is also end-incompressible, and contains no spheres or ∂-parallel disks (by construction), so it is 'essential'. By the Theorem it then either consists of meridional disks, or contains a vertical sublamination w.r.t. some Seifert-fibering of M_i (not necessarily the one that it inherits from M).

Let us consider first the case that $\mathcal{L}_i$ consists of meridional disks. We wish to show that, by an isotopy of $\mathcal{L}$ which controls the intersection of $\mathcal{L}$ with S, we can make $\mathcal{L}$ meet M_i in a collection of taut disks (meaning each disk meets each annulus of $\partial M_i|S$ in essential arcs). To do this, consider $\lambda=\partial\mathcal{L}_i\subseteq\partial M_i$, and its intersection with each annulus complement A_j of S in ∂M_i. This intersection consists of a finite number of parallel families of essential and trivial arcs in A_j.

Note that because λ is (assumed to be) carried by a train track $\tau = B\cap\partial M_i$, there is an upper bound on the number of times a loop in λ can meet S (the loops fall into a finite number of loops parallel in τ; each loop in a family meets S the same number of times). Now, any collection of trivial arcs in an A_j can be removed by an isotopy of $\mathcal{L}$ supported in a neighborhood of the disk which the innermost arc of the family splits off from A_j. This reduces the number of times the loops of λ containing these arcs meets S. By an inductive use of this process, eventually every loop of λ must be taut. Note that this isotopy never adds points to $\mathcal{L}\cap S$, only removes them.

If, on the other hand, $\partial\mathcal{L}_i$ contains non-meridional loops, then $\mathcal{L}_i$ contains an annulus or Möbius band leaf. Look at the collection C of compact leaves of $\mathcal{L}_i$; it is a (closed) sublamination of $\mathcal{L}_i$. $C\cap\partial M_i$ consists of a collection of parallel loops in ∂M_i; by a process similar to that just described, we can make these loops meet S tautly.

There are now two cases to consider:

Case 1: $\partial C\subseteq\partial M_i$ runs parallel to S (i.e., $C\cap S=\emptyset$), or C contains a Möbius band leaf. Then (see Fact 6 of section 3 for the Möbius band case) we can isotope C (in so doing isotoping $\mathcal{L}$) so that C contains a circle fiber of M. Therefore, possibly after splitting $\mathcal{L}$ along the leaf containing the fiber, we may assume that $\mathcal{L}$ misses a circle fiber γ of M, and therefore misses a small (fibered) neighborhood of γ, and so we can consider $\mathcal{L} \subseteq M \setminus \text{int}(N(\gamma)) = M_0$. Now, thought of in M_0, $\mathcal{L}$ is still essential: π_1-injectivity of leaves follows from the injectivity of the composition $\pi_1(L) \to \pi_1(M_0) \to \pi_1(M)$, ∂-injectivity is vacuous ($\mathcal{L}$ misses ∂M_0), irreducibility of $M_0|\mathcal{L}$ follows because γ is essential in $M|\mathcal{L}$, and end-incompressibility follows easily (because M_0 is a codimension-0 submanifold of M).

Therefore by Theorem 4.1, $\mathcal{L}$ contains a vertical sublamination $\mathcal{L}_0$ in M_0, and hence contains a vertical sublamination in M. We had to split $\mathcal{L}$ open to find this sublamination; we need to show that $\mathcal{L}$ also contains a vertical sublamination <u>before</u> splitting.

Consider the component N of $M|\mathcal{L}$ created by the splitting. It is a (possibly non-compact) I-bundle, and it has one or two boundary components which are leaves of $\mathcal{L}$. It is easy to see that they are contained in the vertical sublamination of $\mathcal{L}$ (they are the first leaves that the vertical annuli would meet travelling away from ∂M_0, so the leaves contain vertical loops). Therefore N is saturated by circle fibers, so it is a Seifert-fibered I-bundle, with vertical ∂I-subbundle. It is easy to see that such a bundle has a vertical section L (since N is orientable, there are only 4 cases); but by collapsing N onto L, we reverse the splitting, retrieving our original lamination $\mathcal{L}$ with the vertical sublamination $(\mathcal{L}_0 \setminus \partial N) \cup L$.

Case 2: $\partial C \subseteq \partial M_i$ meets S, and C does not contain a Möbius band leaf. Then every leaf of C is a ∂-parallel annulus, and the loops of ∂C meet S non-trivially and tautly. The leaves of C again fall into a finite number of parallel families in M_i. Choose an innermost leaf L of an outermost family in C, and choose a ∂-compressing disk Δ for L, $\partial\Delta = \alpha \cup \delta$, with $\alpha \subseteq L$ and δ contained in a loop of S. By the usual methods we can assume that $\mathcal{L}$ meets Δ transversely in a collection of arcs.

Figure 8: killing annuli

Then by doing a ∂-surgery on $\mathcal{L}$ using (a disk slightly larger than) Δ, we can split the annulus leaves in the same family as L into a collection of trivial disks (see Figure 8), which we can then isotope away using our previous methods. Note that this creates no new families of annuli or Möbius bands; the effect of surgery on leaves near L is to cut off half-infinite rectangular tails from simply-connected leaves (each parallel family is open and closed in $\mathcal{L}_i$), and cut them into trivial disks. So simply-connected leaves remain simply-connected. It also adds no new points of intersection to S.

After a finite number of such surgeries, we can kill off all of the annulus leaves of C; $\mathcal{L}\cap M_i$ then must consist of meridional disks (because it is still π_1-injective and end-incompressible), which we treat as before.

The construction above forms the core of our isotopy process. Starting with $\mathcal{L}$, either it contains a vertical sublamination or there is a conservative isotopy I_1 so that $I_1(\mathcal{L})$ meets M_1 in a collection of taut disks. We now continue cyclically through our list of solid tori $M_1, \ldots, M_r$, so that at stage j, we are adding to the previous isotopies, trying to make $I_j(\mathcal{L})$ meet M_i in taut disks, where $j \equiv i (\mathrm{mod}\ r)$. By the above construction, either this isotopy can be built, or $\mathcal{L}$ contains a vertical sublamination.

If we therefore assume that $\mathcal{L}$ does not contain a vertical sublamination, then are able to construct an infinite sequence of isotopies I_j with the property that $I_j(\mathcal{L})$ meets M_i in a collection of taut disks. If at any stage $I_j(\mathcal{L})$ meets all of the solid tori $M_1, \ldots, M_r$ in taut disks, then as in section 4 these disks can be 'straightened' out, completing the isotopy of $\mathcal{L}$ to a horizontal lamination. Thus $\mathcal{L}$ is itself a horizontal lamination.

Because each of the above two situations justify the theorem, we (can and) will assume from now on that neither of them hold; i.e. $\mathcal{L}$ does not contain a vertical sublamination, and is not itself isotopic to a horizontal lamination. We will therefore think of these isotopies as defining an infinite isotopy process; we find ourselves forever pushing $\mathcal{L}$ around, and are 'not quite' able to make it all horizontal.

We will need a little more notation to continue. We have defined I_j as the composition of the first j isotopies of $\mathcal{L}$, making $\mathcal{L}$ meet the solid tori M_i cyclically in taut disks. We will let $I_{(j)}$ represent any stage of the isotopy between I_{j-1} and I_j. We will also let $I_{j,k}$ denote the composition $I_k \circ I_j^{-1}$ (i.e., the composition of the isotopies built between the j^{th} and the k^{th} stages), so that $I_{j,k} \circ I_j = I_k$.

b. Finding stable arcs

Now we have an isotopy process, and we assume that it continues indefinitely. This means that at no stage does it succeed in pulling $\mathcal{L}$ horizontal, but for all j, the isotopy I_j succeeds in making $\mathcal{L}$ meet M_i in taut disks, where $j \equiv i \pmod r$. Now for each j, the points $I_j(\mathcal{L}) \cap S$ form a (closed) collection of points in S, the set of sentinel fibers of our Seifert-fibering. By the construction of the isotopy I_j, these points were never moved by any of the isotopies that went into the construction of I_j, i.e., they are stable under these isotopies. In particular, for $j \leq k$, $I_j(\mathcal{L}) \cap S \supseteq I_k(\mathcal{L}) \cap S$, i.e., these sets are nested. They are also non-empty; if $I_j(\mathcal{L}) \cap S = \emptyset$, then $\mathcal{L}$ misses a fiber of M (i.e., any of those in S), and so, by Theorem 4.1, contains a vertical sublamination. But we have assumed $\mathcal{L}$ contains no such sublamination.

So we have a nested sequence of closed, non-empty subsets of the compact set S; their intersection $\cap(I_j(\mathcal{L}) \cap S) = P_0$ is therefore non-empty. P_0 in fact meets every component of S (for otherwise $I_j(\mathcal{L})$ must have missed that component for some j, allowing us to find a vertical sublamination again). By construction, P_0 consists of all of the points of $\mathcal{L}\cap S$ which are never moved by any of the isotopies in our isotopy process, i.e., they represent the stable points of our isotopy process. What we will now show is that, as we watch the isotopies progress, these points become 'islands of stability' for the process; a stable (horizontal) lamination starts to 'grow' out of them.

Now, consider a 1-simplex $e_i \in B^{(1)}$ and the annulus $A_i=\pi^{-1}(e_i)$, $\partial A_i \subseteq S$. Pick points x, x' of P_0, one in each component of ∂A_i. What we wish to look at now are the arcs of $I_j(\mathcal{L})\cap A_i$ containing x, x' (call them, respectively, α, α'), and how they change under further isotopies. Because for each arc one of its endpoints is anchored down (x, x' are stable), the only way these arcs can change is by 'boundary compressions' (see Figure 9). Our intent is to show that for some $k\geq j$, each of these arcs $I_{j,k}(\alpha)$, $I_{j,k}(\alpha')$ has both of its endpoints in P_0. This arc would therefore be stable, i.e., $I_{j,k}(\alpha)$ (say) would be fixed under all further isotopies.

We proceed as follows. Given α, $\alpha'\subseteq A_i$, there exists an arc ω_j (for 'winding number') joining x to x' and not meeting α, α' except at their endpoints. This is because A_i split on α, α' has 2 or 3 (if α, α' are both trivial arcs in A_i) components, at least one of which contains both x and x'.

Lemma 5.1: If at some further stage $I_{j'}$ of the isotopy process, one of the arcs emanating from x, x' has non-zero winding number wrt. ω_j (meaning it is not isotopic rel endpoints to an arc meeting ω_j only at its endpoints), then at some stage of the isotopy process between I_j and $I_{j'}$, one of the arcs emanating from x or x' was trivial, i.e., ∂-parallel in A_i.

Proof: Since α, α', have zero winding number wrt. ω_j, and change only by ∂-compressions, there is a first ∂-compression after which one of the arcs has non-zero winding number. We claim that, at the time of this compression, one of the arcs is trivial.

For suppose not; note that since the stable ends of the arcs α, α' are on opposite sides of $\mathrm{A_i}$, the ∂-compression leaves one of the arcs, say α', fixed. Since this is the first ∂-compression where the winding number changes, we have that the winding number of α' is zero. Now if α' is not trivial, then its other endpoint is on the same side as x (see Figure 9). Since α is not trivial, its other endpoint is on the x'-side of $\mathrm{A_i}$, so the ∂-compression is taking place on that side. But because after the compression the arc emanating from x cannot meet α' (because after the compression, $\mathcal{L}$ still meets $\mathrm{A_i}$ in a lamination, which can't have leaves intersecting), which hasn't been moved, only one of two things can have occurred: either **(1)** the new arc α_{new} is a trivial arc, in which case it is isotopic rel endpoints to an arc in $\partial \mathrm{A_i}$, with x as an endpoint, so has zero winding number, or **(2)** α_{new} is an essential arc (which lies in $\mathrm{A_i}|\alpha'$, which is a disk), and so is isotopic rel x to α, by an boundary-preserving isotopy which does not meet x'; and therefore α_{new} also has winding number zero w.r.t. ω_j, since it must then have the same winding number that α has. Both of these situations, however, violate our hypothesis, giving the necessary contradiction. ∎

Figure 9: winding numbers

In other words, if one of the arcs moves alot, then one of the arcs had to be trivial (at some time).

It then follows, by an inductive use of the lemma (since the arcs emanating from x, x' in $\mathrm{I_{nr+i}}(\mathcal{L}) \cap \mathrm{A_i}$ are non-trivial (they are contained in the boundary of taut disks in $\mathrm{M_i}$)), that one of two things will happen:

(1) one of the points x, x', is the endpoint of a trivial arc in $\mathrm{I_{(k)}}(\mathcal{L}) \cap \mathrm{A_i}$ infinitely often (i.e., for arbitrarily large values of k),

or

(2) eventually, neither point is ever contained in a trivial arc, and there exists j, and ω_j so that for k$\geq$j, the arcs of $\mathrm{I_{(k)}}(\mathcal{L}) \cap \mathrm{A_i}$ emanating for x, x', never have non-zero winding number wrt. ω_j.

What we now show is that the first of these possibilities must necessarily lead to a contradiction, while the second leads to the eventual stability of the arcs emanating from x, x' (in order to avoid a contradiction similar to the one encountered in the first case).

First case: x (say) is contained in a trivial arc α_k of $I_{(k)}(\mathcal{L}) \cap \mathrm{A_i}$ for arbitrarily large values of k.

What we will do now is watch the proliferation of the intersections of these trivial arcs with γ, a loop in $\mathrm{A_i}$ lying parallel to the component of $\partial \mathrm{A_i}$ containing x. Recall that our isotopies are conservative, so that the only points of the intersection of $\mathcal{L}$ with the sentinel fibers which move are those which disappear. Now the effect of a ∂-compression on the arc α_k is to cut off a short arc near its non-stable end, and splice it to another arc by an arc running in the annulus between γ and the loop of $\partial \mathrm{A_i}$ it runs next to. We may assume that such compressions do not remove points of intersection of α_k with γ. We can therefore assume that the points of $\alpha_k \cap \gamma$ are <u>fixed</u> under all further isotopies, i.e., $\alpha_k \cap \gamma \subseteq \alpha_{k'} \cap \gamma$ whenever $k' \geq k$. Since the arc containing x periodically becomes essential (every time $\mathcal{L} \cap M_i$ is pulled taut), it follows that this inclusion is usually proper, i.e., these trivial arcs continue to pick up more and more points of intersection with the neighbor loop as k gets larger and larger. It is the fact that these points must be piling up on one another in the neighbor loop that is going to give us our contradiction.

First we need some notation. Let ω be an essential arc in $\mathrm{A_i}$ whose endpoints in $\partial\mathrm{A_i}$ are not in $\mathcal{L}$ (in fact, since $\mathcal{L}\cap\partial\mathrm{A_i}$ is closed, we may assume ϵ -neighborhoods (in ∂A_i) of the endpoints do not meet $\mathcal{L}$, for sufficiently small ϵ). Orient ω with tail z on the component S of $\partial\mathrm{A_i}$ containing x. z and x separate S into two arcs, called the left side and the right side of ω. Orient the α_k with tail at x, and orient the neighbor loop γ. Using these orientations, we can assign local orientations to the points of $\alpha_k\cap\gamma$, and winding numbers to the arcs of α_k between x and a point of $\alpha_k\cap\gamma$. Note that because the isotopies are constant near the points of $\alpha_k\cap\gamma$, and $(\alpha_k\cap\gamma) \subseteq (\alpha_{k+1} \cap \gamma)$, it follows that the local orientation assigned to a point is the same as the one assigned when thought of as living in every further arc α_k. Also, the winding numbers associated to a subarc of α_k is actually a function of its endpoint $t \in\alpha_k\cap\gamma$, because the arcs in α_k and in α_{k+1} between x and t are identical.

Call the other endpoint of α_k (i.e. the one which isn't x) x_k, and the intersection point of α_k with γ, which is adjacent to x_k along α_k, y_k (see Figure 10).

Figure 10: stabilization: first case

Now, the winding number of the arc β_i of α_i between x and y_i is always either -1, 0, or 1. This is because β_i differs from α_i only in the short arc between x_i and y_i

(which doesn't meet ω), and α_i has one of the above mentioned winding numbers because, being trivial, it is homotopic (in fact isotopic) rel endpoints to a subarc of S$\in \partial$A$_i$, which meets the winding arc ω at most once. Therefore, the winding numbers assigned to the points y_i in α_k are either -1, 0, or 1.

Now lift the α_i to the universal cover π:R$\times I \to$ A$_i$ of A$_i$, sending x to (0,0)=$\tilde{x}$, and let $\tilde{y}_i$ be the resulting lifts of the points y_i, obtained by lifting the α_i. Let $\tilde{\gamma}$=$\pi^{-1}(\gamma)$, so $y_i \in \tilde{\gamma}$. Because we could calculate the winding number of α_i w.r.t. ω by lifting α_i to $\tilde{\alpha}_i$ and count the winding number w.r.t. <u>all</u> of the lifts of ω in $\mathbf{R}\times$I, and this amounts basically to calculating the integer part of the first coordinate of $\tilde{y}_i \in \mathbf{R}\times$I, it follows that the points $\tilde{y}_i$ must lie in a compact piece [-2,2]$\times$I of $\mathbf{R}\times$I.

So these points $\tilde{y}_i$ must be piling up on one another. In particular, for any $\epsilon > 0$, there exist points $\tilde{y}_i$, $\tilde{y}_j$, $j < i$, which are within ϵ of one another along $\tilde{\gamma}$. $\tilde{\beta}_i \setminus \tilde{\beta}_j$ is the arc of $\tilde{\alpha}_i$ between $\tilde{y}_j$ and $\tilde{y}_i$, which together with the arc of $\tilde{\gamma}$ between these two points, forms a (null-homotopic; $\mathbf{R}\times$I is contractible) loop in $\tilde{A}_i$. This loop projects down in A$_i$ to a loop consisting of the arc β=$\beta_i \setminus \beta_j$ in α_i, together with an arc of length $< \epsilon$ in γ, and this loop is null-homotopic.

Now, consider this short arc δ between y_i and y_j in γ. If $\beta \cap \delta \subseteq \partial\delta$, then $\beta \cup \delta$ is an embedded null-homotopic loop in A$_i$, hence bounds a disk D in A$_i$ with ∂D= $\beta \cup \delta$, where $\beta \subseteq \mathcal{L}$, and δ is a short arc (of length $< \epsilon$) transverse to $\mathcal{L}$. But looking back across the isotopies carried out so far, this disk demonstrates a homotopy of a vertical arc in N(B), rel its boundary, into a leaf of $\mathcal{L}$. This, however, contradicts [G-O, Theorem 1(d)], which says that such homotopies are impossible.

If β meets δ in the interior of δ, then since $\beta \subseteq \alpha_i$, it follows that α_i meets δ in interior points. Now α_i cuts off a disk Δ in A$_i$; think of it as being colored <u>green</u>. Δ meets γ in subarcs of $\gamma \setminus \alpha_i$; think of these as being colored green as well.

Because α_i separates $\mathrm{A_i}$, it follows that $\gamma\backslash\alpha_i$ consists of an even number of arcs, which (travelling along γ) are alternately colored green and left uncolored (locally, α_i is colored green on only one side).

Since α_i meets $\delta \subseteq \gamma$ in interior points, it follows that $\delta\backslash\alpha_i \subseteq \gamma\backslash\ \alpha_i$ contains a <u>colored</u> <u>subarc</u>, δ_0. δ_0 is contained in Δ, properly embedded, and so it splits Δ into two disks, one of which, Δ_0, does not contain the arc $\eta = \partial\Delta \cap \partial\mathrm{A_i}$. Therefore, $\partial\Delta_0 = \delta_0 \cup \alpha_0$, with $\alpha_0 \subseteq\alpha_i\subseteq\mathcal{L}$, and $\delta_0 \subseteq \gamma$, transverse to $\mathcal{L}$, with length < the length of $\delta < \epsilon$. This, however, once again contradicts [G-O, Theorem 1(d)].

Therefore, this first situation is impossible.

Second case: α_i and α_i' are always essential (for $i > i_0$), and there is some essential arc $\omega \subseteq\mathrm{A_i}$ joining x and x' so that α_i and α_i' always have winding number zero w.r.t. ω.

We wish now to show that eventually α_i (say) becomes <u>stable</u>, i.e., for some i, α_k=α_i, for all $k \geq i$. This amounts to saying that $x_k = x_i$, for all $k \geq i$, i.e., $x_i \in\mathrm{P_0}$.

So assume the contrary; assume that $x_{k_i} \neq x_{k_{i-1}}$, for $k_i > k_{i-1}$, infinitely often (to save the reader's eyesight, we will conveniently forget that this expression has a 'k' in it, and write x_i instead). We will then obtain a contradiction, in a manner similar to the first case (with some slight technical additions).

We get an arbitrarily large collection of distinct points $y_i\in \gamma$, i= $1, 2, \ldots$, in the α_i which are near neighbors to the endpoints x_i of the α_i. Now, as before, we can lift the α_i, α_i' to $\mathbf{R} \times \mathrm{I} =\tilde{\mathrm{A}}_\mathrm{i}$, with $\tilde{x}$ =(a,0), $\tilde{x}'$ =(b,1) fixed. Because the winding number of the lifts of α_i can be counted across the lifts of ω, it follows that the endpoints $\tilde{x}_i$ of the lifts of the α_i based at $\tilde{x}$ all lie in the interval [b-1,b+1]$\times$1 and so the points y_i are contained in a compact piece ([b-1,b+1]$\times$I)$\cap\tilde{\gamma}\subseteq\tilde{\gamma}$ of the neighbor line on the $\tilde{x}_i'$-side of $\mathbf{R}\times$I. So as before we have an arbitrarily large number of $\tilde{y}_i$

accumulating in a fixed compact piece of $\tilde{\gamma}$, so eventually we can find (adjacent) points of (some) $\tilde{\alpha}_i \cap \tilde{\gamma}$ which are within ϵ of one another. the arc of $\tilde{\alpha}_i$ joining these two points, together with the arc of $\tilde{\gamma}$ joining them, form an (embedded) loop in $\mathbf{R}\times$I, which descends to a (singular) null-homotopic loop in A_i.

Lemma 5.2: If we orient α_i, α_i' so that x, x' are at their tails, and look at the normal orientations that this induces on the set T= $(\alpha_i \cap \gamma) \cup (\alpha_i' \cap \gamma)$ of (transverse) intersection points with γ, then seen from γ they occur with opposite sign.

Proof: α_i and α_i' together separate A_i (although each separately doesn't) into two disks D_1, D_2 (see Figure 11), with the orientations of α_i, α_i', giving orientations two two arcs in each boundary, as shown. Any arc δ of γ between two adjacent points of T must lie in either D_1 or D_2 (D_1, say). If the endpoints of δ both lie on the same end of ∂D_1 then measured along δ the normal orientations of its endpoints are opposite; if they lie on opposite ends of ∂D_1, then, because we chose the orientations of α_i and α_i' to complement one another as they do, measured along δ the normal orientations of its endpoints are again opposite. ∎

Figure 11: normal orientations

Note that this lemma would not be true if we dealt with only one arc (α_i, say) at a time; this is because by itself α_i, say, does not separate A_i (see Figure 11). Note also that if we lift α_i to $\tilde{\alpha}_i$ in $\mathbf{R}\times$I, with the lifted orientation, and look at the normal orientations with which $\tilde{\alpha}_i$ meets the neighbor line $\tilde{\gamma}$, as you travel along

$\tilde{\gamma}$ these also alternate; this is because $\tilde{\alpha}_i$ now does separate $\mathbf{R}\times$I, so the situation is just as in the first case of the lemma above.

Now, we have already found adjacent points of (some) $\tilde{\alpha}_i \cap \tilde{\gamma}$ which are within ϵ of one another along $\tilde{\gamma}$. By the note above, these two points inherit opposite normal orientations in $\tilde{\gamma}$ from $\tilde{\alpha}_i$. Together with the arc of $\tilde{\gamma}$ between them, the arc of $\tilde{\alpha}_i$ joining them forms an embedded null-homotopic loop in $\tilde{\mathrm{A}}_{\mathrm{i}}$, which descends to a null-homotopic loop in A_{i}, consisting of an arc β of α_i between points y_{i_o} and y_{i_1} of $\alpha_i \cap \gamma$, together with the short arc δ of γ between them. If $\beta \cap \delta = \partial\delta$, then, as before, $\beta \cup \delta$ is an embedded null-homotopic loop; the disk it bounds gives a null-homotopy violating [G-O, Theorem 1(d)], a contradiction.

If $\beta \cap \delta \neq \partial\delta$, then in particular $\alpha_i \cup \alpha_i'$ meets δ in interior points. Now these points of intersection inherit normal orientations from α_i and α_i', which when seen along δ occur with opposite sign. The endpoints of δ also have opposite sign (their lifts did in $\tilde{\gamma}$, and they remain the same when projected); it then follows that there are an even number of points in C=$(\alpha_i \cup \alpha_i') \cap \delta$. Since the endpoints of δ both belong to α_i, it then also follows that some pair of points of C, adjacent along δ, both belong to α_i or α_i' (say α_i), joined by a subarc δ_0 of δ. Now α_i and α_i' together separate A_{i} into two disks D_1 and D_2, and since δ_0 doesn't meet α_i or α_i' except at its endpoints, δ_0 is contained in one of these disks, say D_1. δ_0 separates this disk into two sub-disks; because both of the endpoints of δ_0 are in α_i, one of these disks Δ_0 does not meet $\partial\mathrm{A}_{\mathrm{i}}$ (see Figure 12), so its boundary $\partial\Delta_0 = \delta_0 \cup \beta_0$, where β_0 is a subarc of α_i. This disk Δ_0 would again give a homotopy violating [G-O, Theorem 1(d)], and so gives a contradiction.

Figure 12: stabilization: second case

So all other possibilities lead us to a violation of [G-O, Theorem 1(d)]; we must therefore conclude that, eventually, the arcs α_i, α_i', for some i, emanating from the points x,$x'$$\in$$\mathrm{P}_0$ are stable: their other endpoints are also in P_0.

c. Proof of the theorem

We are now in a position to complete the proof of the theorem.

Given a point $x \in \mathrm{P}_0$ in the stable set of our isotopy process, and an annulus $\mathrm{A_i}$ containing it, in the boundary of a solid torus $\mathrm{M_i}$, we have shown that for some j, the arc α_j of $\mathrm{I_j}(\mathcal{L})\cap\mathrm{A_i}$ which contains x is stable; all further isotopies of $\mathcal{L}$ fix α_j. This is equivalent to saying that its other endpoint is also in P_0; since such an arc would only be changed by ∂-compressions, and both its endpoints are stable, this means that the arc cannot be moved by further isotopies.

Lemma 5.3: Given $x \in \mathrm{P}_0$, there is a neighborhood $\mathcal{U}$ of x in S and a j so that for any $x' \in \mathcal{U}\cap\mathrm{P}_0$ and $\mathrm{A_i} \subseteq\partial\mathrm{M_i}$ containing x', x' is contained in a stable arc of $\mathrm{I_j}(\mathcal{L})\cap\mathrm{A_i}$.

Proof: Fix an annulus $\mathrm{A_i}$ containing x. By the above, there is a j so that x is contained in stable arc α of $\mathrm{I_j}(\mathcal{L})\cap\mathrm{A_i}$, with other endpoint x'. Let $\mathcal{U}$ be a (closed) ϵ-neighborhood of x in the loop of S containing x, intersected with P_0, and consider the (taut) arcs of some $\mathrm{I_{kn+i}}(\mathcal{L})\cap\mathrm{A_i}$, with $\mathrm{kn+i \geq j}$, emanating from these points (then set $\mathrm{j = kn + i}$). P_0 is closed, so $\mathrm{P}_0\cap\mathcal{U}$ is closed in $\mathcal{U}$; there is therefore a highest and lowest point of P_0 in $\mathcal{U}$. By choosing a larger j, if necessary, we may additionally assume that the arcs of $\mathrm{I_j}(\mathcal{L})\cap\mathrm{A_i}$ emanating from these points are also stable. The collection $\mathrm{I_j}(\mathcal{L})\cap\mathrm{A_i}$ of arcs is a 1-dimensional lamination in $\mathrm{A_i}$, which are all parallel to one another.

Now, suppose an arc β of $\mathrm{I_j}(\mathcal{L})\cap\mathrm{A_i}$ emanating from a point in $\mathrm{P_0}\cap\mathcal{U}$ moves under a further isotopy. Consider the first time such a move occurs. Because the endpoint of the arc on the x-side is stable, the change occurs as a ∂-compression on the x'-side of $\mathrm{A_i}$.

If the resulting arc is trivial, then because the points at either end of $\mathcal{U}$ are in stable (essential) arcs, the disk that it cuts off of $\mathrm{A_i}$ therefore meets $\partial\mathrm{A_i}$ in an arc of $\mathcal{U}\backslash x$ (because x is contained in a stable arc, too), which therefore has length less than ϵ.

If the resulting arc is still essential, then the ∂-compression joined β to a trivial arc on the x'-side of $\mathrm{A_i}$. But such a trivial arc (since all of the arcs between the highest and lowest (essential) arcs emanating out of $\mathcal{U}$ were essential at stage j) had to be created by some x-side ∂-compression at some stage after k; this trivial arc (immediately after the compression) had to meet the neighbor loop γ on the x-side, and a subarc, together with a short arc of γ (of length $< \epsilon$), bounds a disk in $\mathrm{A_i}$.

In each case we therefore have a situation which gives a disk violating [G-O,Theorem 1(d)], a contradiction.

Repeating this argument for each of the annuli containing x, taking the maximum of the j's generated and the intersection of the $\mathcal{U}$'s generated, completes the proof. ∎

Now we have that for each x in $\mathrm{P_0}$ there exists a pair $(\mathcal{U}_x, j_x)$ given by the lemma. The collection of $\mathcal{U}_x$'s form an open cover of $\mathrm{P_0}$, which, because it is compact ($\mathrm{P_0}$ is closed in S, which is compact), has a finite subcover, $\{\mathcal{U}_1, \ldots, \mathcal{U}_n\}$. Set j=max$\{j_1, \ldots, j_n\}$, then it follows that every arc of $\mathrm{I_j}(\mathcal{L})\cap\mathrm{A_i}$ emanating from any point of $\mathrm{P_0}$, for any $\mathrm{A_i}$, is stable; it has both of its endpoints in $\mathrm{P_0}$.

Now choose a point $x \in P_0 \cap M_i$, for any given M_i. For some r, $0 \leq r < n$, x is contained in a taut disk D of $I_{j+r}(\mathcal{L}) \cap M_i$. But by dragging ourselves around ∂D starting from x, we see inductively (using the above) that every point of $\partial D \cap S$ is in fact contained in P_0, i.e., the boundary of this disk is stable, and therefore the disk containing x is stable. It therefore follows that for every $x \in P_0$, and every M_i containing x, x is contained in a stable, taut, disk of $I_{j+n}(\mathcal{L}) \cap M_i$. Because $P_0 \cap M_i$ is a closed set in ∂M_i, it follows that the collection of disks of $I_{j+n}(\mathcal{L}) \cap M_i$ containing points of P_0 is a (closed) sublamination of $I_{j+n}(\mathcal{L}) \cap M_i$; the union of these disks over all of the M_i then forms a sublamination $\mathcal{L}_0$ of $I_{j+n}(\mathcal{L})$ (they meet correctly along the ∂M_i, in the (stable) arcs emanating from P_0), which meets each M_i in a collection of tauts disks. By a small further isotopy of $I_{j+n}(\mathcal{L})$ (first supported in a neighborhood of the ∂M_i to make the boundaries of the taut disks transverse to the circle fibering of ∂M_i, then supported away from ∂M_i to make the entire disks transverse) we can make $\mathcal{L}_0$ into a lamination meeting each solid torus in a collection of <u>transverse</u> disks, i.e., $\mathcal{L}_0$ is a horizontal lamination.

Therefore, $\mathcal{L}$ contains a sublamination $I_{j+n}^{-1}(\mathcal{L}_0)$ which is isotopic to a horizontal lamination.

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