Free Seifert surfaces and disk decompositions

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Abstract. In this paper we construct families of knots which have genus one free Seifert surfaces which are not disk decomposable.

§0
Introduction

A Seifert surface for a knot K in the 3-sphere is an embedded orientable surface Σ, whose boundary equals the knot K. Equivalently, it is a properly embedded orientable surface Σ in the exterior $X(K)$ of K, whose boundary equals the longitude of K. A Seifert surface Σ is *free* if $\pi_1(S^3 \setminus \Sigma)$ is a free group; equivalently, $S^3\setminus\mathrm{int}N(\Sigma)$ is a handlebody.

Seifert's algorithm [Se] will always build a free Seifert surface for a knot K. In [Br] we showed that not all free Seifert surfaces can be built by Seifert's algorithm; we exhibited a family of hyperbolic knots having *free genus* one, whose surfaces built via Seifert's algorithm must always have large genus.

In so doing, we introduced a fairly general procedure for producing knots with genus 1 free Seifert surfaces. In this paper we show that many of these surfaces fail to be disk decomposable.

A sutured manifold (M,γ) is a compact 3-manifold M together with a collection of disjoint embedded loops γ in ∂M, called the sutures. (Since we will apply this theory to knots and Seifert surfaces, we will suppress the possibility that whole components of ∂M are sutures). The boundary of M can be expressed as $\partial M = R_+(\gamma) \cup R_-(\gamma)$, with $R_+(\gamma) \cap R_-(\gamma) = \gamma$. We give R_+ a transverse orientation pointing into M, and R_- a transverse orientation pointing out of M. We think of each component of γ as having as having a transverse orientation pointing from its R_+ side to its R_- side, . For further details, see [Ga1].

A *decomposing surface* for (M,γ) is a properly embedded, transversely oriented surface F which is transverse to γ. We can , by matching the transverse orientations for R_+, R_-, and F, endow M split open along F, $M|F$, with the structure of a sutured manifold; see Figure 1. The new sutures for $M|F$ are obtained as an 'oriented sum' of γ and ∂F. A sequence of such splittings is called a *sutured manifold decomposition* of (M,γ).

Key words and phrases. hyperbolic knot, free genus, disk decomposition.

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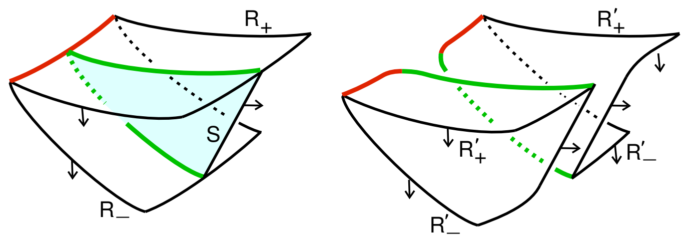

Figure 1

A Seifert surface Σ is *disk decomposable* if the sutured manifold

$$(S^3 \backslash \mathrm{int} N(\Sigma), \Sigma \cap \partial N(K)) = (X_F, \gamma)$$

admits a sutured manifold decomposition whose decomposing surfaces are all disks, ending with a sutured manifold which is the disjoint union of sutured manifolds of the form (B^3, e), where e is the equatorial circle of the 3-ball B^3. (A posteriori, $S^3 \backslash \mathrm{int} N(\Sigma)$ is a handlebody, since it may be cut open along disks to 3-balls, so Σ is free.) By Gabai [Ga2], if Σ is disk decomposable, then the corresponding sutured manifold is *taut*, and so in particular Σ has minimal genus among all surfaces representing its homology class. In other words, the genus of Σ equals the genus of K.

Disk decomposability therefore gives an effective way to compute the genus of a knot. For example, Gabai [Ga2] has shown that every knot in the standard tables [Ro] has a projection for which Seifert's algorithm gives a disk decomposable surface. A fairly natural natural question to ask, then, is: how can we tell, short of producing a set of decomposing disks, that a Seifert surface is disk decomposable? Our main result shows that being free and having minimal genus, which are necessary, are not sufficient.

Theorem. *There exist knots K in S^3 which admit genus one incompressible free Seifert surfaces which are not disk decomposable.*

Our result leaves open the question of whether or not these knots admit *other* Seifert surfaces which are disk decomposable; we discuss this possibility in the concluding section of the paper.

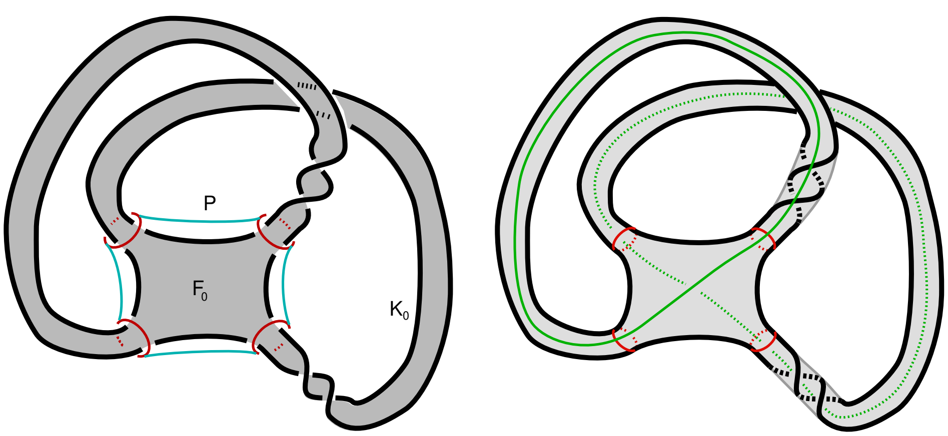

Figure 2

§1
Building free Seifert surfaces

In [Br] we showed that, for the knot K_0 and the free Seifert surface F_0 for K_0 in S^3, shown in Figure 2, $1/n$ Dehn surgery on any loop L in the 4-punctured sphere P pictured there will essentially re-embed K_0 and F_0 as a new knot K and free Seifert surface F in S^3. (There is, in fact, nothing special about this knot; any free Seifert surface for a knot admits similar 4-punctured spheres.) What we will show now is that for appropriate choices of L and n, F will be incompressible but not disk decomposable. Our sutured manifold $X_F = S^3 \setminus \mathrm{int} N(F)$ will be a genus-2 handlebody, and the suture will be a loop $\gamma = F \cap \partial X(K)$ which splits ∂X_F into two once-punctured tori. The essential idea is that if γ is complicated enough with respect to a set of cutting disks for X_F, then F must be incompressible in $X(K)$, but not disk decomposable.

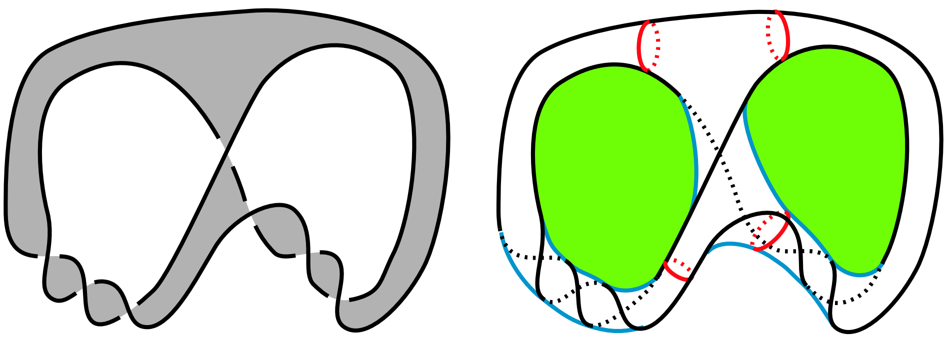

Figure 3

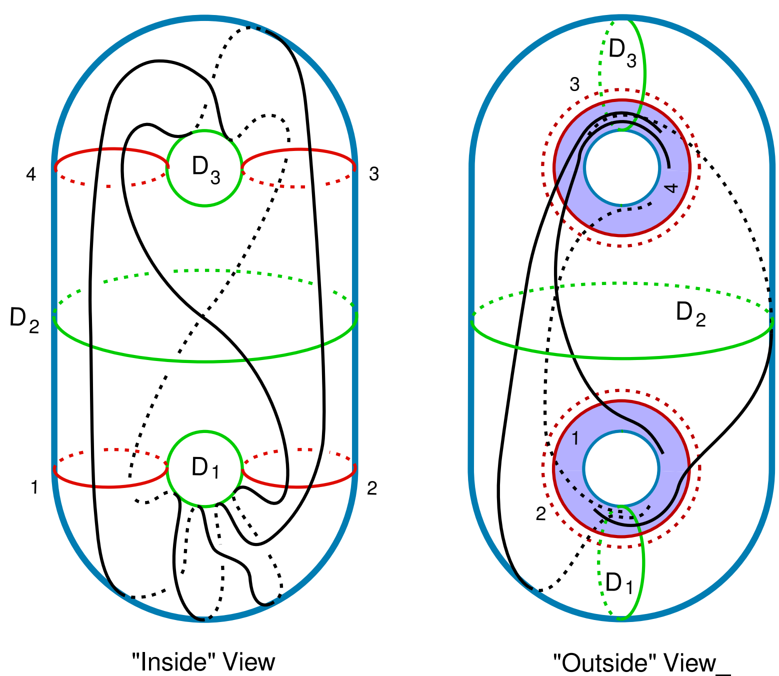

Figure 4

Our argument will be based on techniques of Goda [Go], who first showed that there exist taut sutured handlebodies (H,γ) which are not disk decomposable. Our main task will be to show that his arguments can be applied to some of the sutured handlebodies built as in Figure 2. Because Goda's techniques use the standard view of a handlebody, as the inside of a standardly embedded genus two surface, we first need to produce an 'external' view of X_{F_0}. In other words, we need to understand what our suture γ looks like when X_{F_0} is pictured as the interior of a standardly embedded handlebody in S^3. This involves, essentially, determining what the two annuli in ∂X_{F_0} that are cut off by ∂P would look like on a standard handlebody, while keeping track of the pattern of intersections of γ and ∂P with a set of three 'obvious' cutting disks for X_{F_0}, whose boundaries are shown in Figures 3 and 4. This pattern determines γ up to homeomorphism, in fact, up to Dehn twists along the three cutting disks, since these disks form a complete system of cutting disks for X_{F_0}.

This change of viewpoint is carried out in Figure 4. Our suture γ can be thought of as four arcs lying on a 4-punctured sphere (essentially, P) in X_{F_0}, together with two pairs of arcs spiralling through the complementary annuli in ∂X_{F_0} . The amount of spiralling is determined by how many full twists we put in each arm of our original Seifert surface F_0, and will not play a large role in our further discussions (although the *direction* of spiralling *is* important).

§2 Choosing loops L

It is easy to see what effect $1/n$ Dehn filling on a loop L in P will have on the picture of our sutured handlebody (X_{F_0},γ) above. The loop L will lie slightly inside of the 4-punctured sphere P lying on ∂X_{F_0}, and the disk D it bounds will lie for the most part outside of the handlebody X_F, since it mostly lies in $N(F_0)$. Inside of our handlebody we will see only an annulus running from L to a parallel loop (L', say) in P. $1/n$ Dehn surgery along L will have the effect of replacing the suture γ with its 'sum' with n parallel oriented copies of the loop L' (Figure 5). It will in fact be the result of applying n Dehn twists in an annular neighborhood of L' to γ. This gives us a large family of sutured handlebodies to work with, each of which is realized as the complement of some genus one free Seifert surface in S^3.

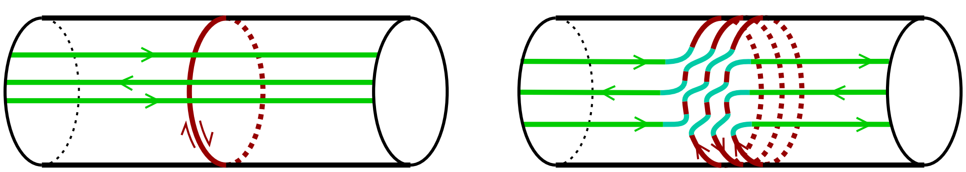

Figure 5

We illustrate this with a somewhat more complicated loop L, in Figure 6; it meets K_0 in 8 points, and so n Dehn twists along L will result in a knot K, which on ∂H will be represented by the 'sum' of K_0 and 8n parallel copies of L. We show the results of one Dehn twist, in Figure 7.

This new suture C meets the standard cutting disks for the handlebody H (which are the three disks where a horizontal plane perpendicular to the paper meets the middle of the figure) only in arcs joining distinct cutting disks. These arcs run, in each of the pairs of pants in ∂H, above and below the cutting disks, between any pair of the cutting disks. It is also easy to see that there are no trivial arcs, running from a cutting disk to itself. This implies that $\partial H \setminus C$ is incompressible in H ([St],[Ko1]), and (H, C) is therefore taut.

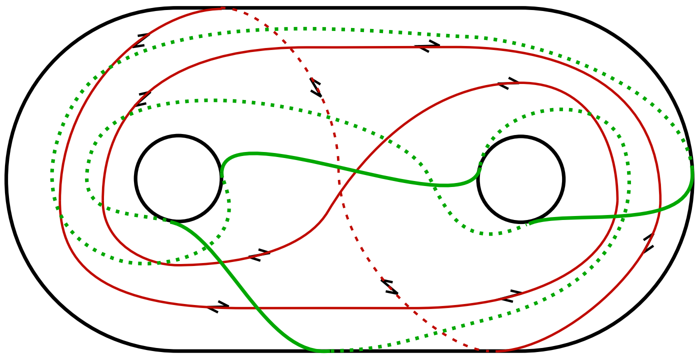

Figure 6

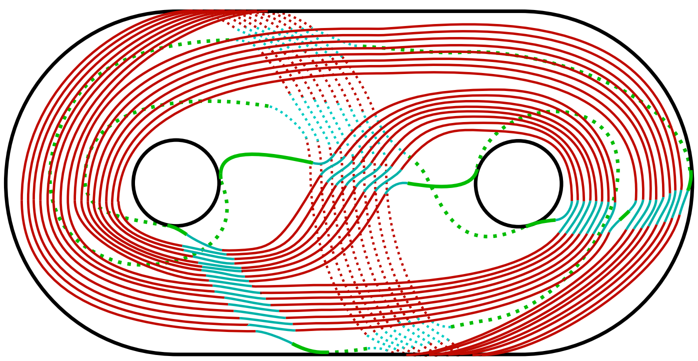

Figure 7

We can 'encode' this construction, and the Dehn twisting information, into a train track τ on ∂H (Figure 8a) carrying both γ, L, and the result C of 'right-handed' Dehn twists of γ along L. This allows us to see, even for a large number of Dehn twists, that all of the loops so built represent sutures of taut sutured handlebodies, since it is easy to see that any loop (which separates ∂H) carried with full support by τ has arcs running between any pair of the cutting disks D_i, on each side, as before, and has no trivial arcs. This is most easily seen by cutting ∂H (and τ) open along our cutting disks (Figure 8b); the resulting train tracks carry no trivial arcs or circles.

This curve L (and the resulting sutures C) will, in the end, still not be sufficiently 'complicated' for our purposes. But several, which will be, share many of the same properties, being carried by the same train track τ.

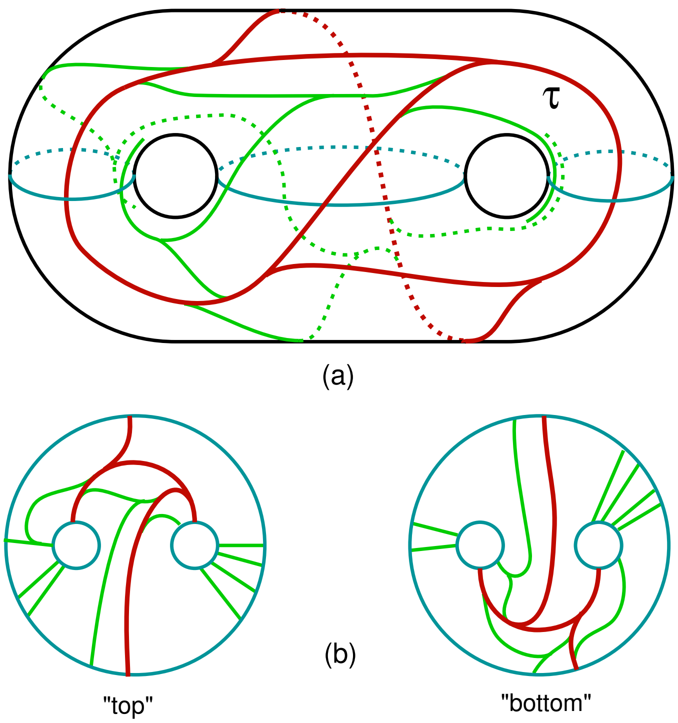

Figure 8

§3 Complicated intersections imply no disk decomposition

Goda [Go] determined sufficient conditions, based on the intersections of the suture C in the boundary of a genus two handlebody H with a system of cutting disks D_1, D_2, D_3 for H, to guarantee that the sutured handlebody (H, C) is taut but not disk decomposable. We will prove here a slightly weaker form of Goda's criterion, which is sufficient for our purposes. Note that any loop C in ∂H locally separates ∂H (i.e., it separates a neighborhood of itself), so we can always unambiguously talk about being on the 'same side' of C in ∂H.

Proposition 1. *If D is a compressing disk for ∂H, with ∂D transverse to C, such that $C|D$ contains three parallel arcs whose ends all lie on the same side of ∂D, then (H, C) is not disk decomposable along D.*

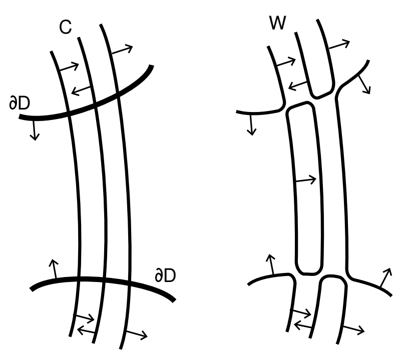

Figure 9

Proof: The picture we have is as in Figure 9. (All other possible choices of normal orientation can be obtained from the one pictured by some combination of changing every orientation or reflecting in a vertical axis, which will not change the essential features of our argument.) Given a transverse orientation on the disk D, the sutured manifold obtained by cutting H along D is one or two solid tori $H|D$, whose sutures are obtained by cutting and pasting C and ∂D near their points of intersection, as in Figure 1. However, because of our hypothesis, the sutures of the resulting sutured solid torus (M, C') will include a component which is null-homotopic in ∂M (Figure 9), and so (M, C') cannot be taut. The key point here is that C separates ∂H, and so the transverse orientations of C, seen along ∂D, must alternate.∎

The reader can note that in Figure 7, each of the disks D_i will have 3 such arcs on each side. We illustrate one such collection in Figure 10.

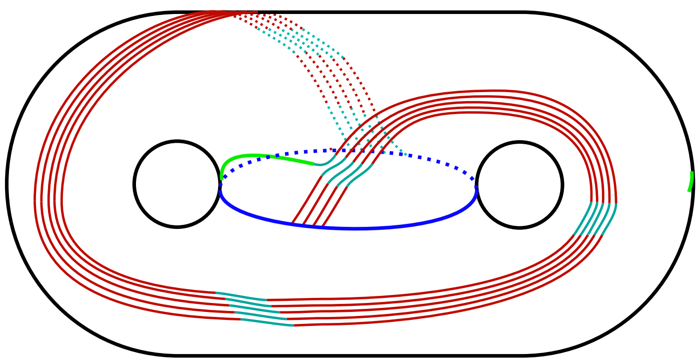

Figure 10

We now assume that C satisfies the conditions thus far introduced: the cutting disks D_i cut C into arcs in the two pairs of pants $\partial H|(\partial D_1 \cup \partial D_2 \cup \partial D_3) = P_1 \cup P_2$. Each arc joins distinct ∂-components in the P_i, and there are arcs running between all possible pairs of ∂-components of the P_i. We also have, for each disk D_i, a set of three parallel arcs in $\partial H|\partial D_i$, as in Proposition 1.

Proposition 2. *Any disk D, isotopic to one of the disks D_1, $i = 1, 2, 3$ and transverse to C, is not a decomposing disk for (H, C).*

Proof: This is essentially Claim 3.6 of [Go]; for completeness, we reproduce the argument here, since many of the same ideas will be used later.

Without loss of generality, we may assume the D is isotopic to D_1; then by [Ep,Lemma 2.5] there is an innermost disk Δ in ∂H whose boundary consists of an arc α of ∂D_1 and an arc β of ∂D. C intersects Δ in arcs, and by our hypothesis, none of these arcs have both endpoints on α. If any have both endpoints on β, then there is an outermost such arc δ; but then it is easy to see that either decomposing (H, C) along D yields a trivial suture, implying the (H, C) is not disk decomposable along D (Figure 11a), or we may isotope C across the outermost disk cut off by δ, without altering the sutured manifold obtained by decomposing along D (Figure 11b). Continuing, we can remove all such trivial intersections of C with D (or obtain our desired conclusion).

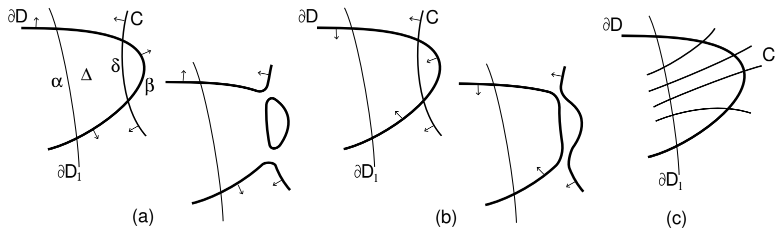

Figure 11

We may therefore assume that C meets Δ only in arcs running from α to β, which must therefore all be parallel to one another (Figure 11c), and so we can isotope ∂D across Δ, removing two points of intersection of ∂D with ∂D_1, without changing the intersections of ∂D with C. Continuing, we can then assume that ∂D and ∂D_1 are disjoint, and so by [Ep,Lemma 2.4] they cobound an annulus B. By the same argument, we may assume that C meets B only in arcs running from ∂D to ∂D_1, and so we may isotope ∂D to ∂D_1 without changing the intersections of ∂D with C. Therefore the sutured manifold resulting from decomposing along D is identical with the one obtained by decomposing along D_1. But by our hypotheses and Proposition 1, (H,C) is not disk decomposable along D_1, and so it cannot be disk decomposable along D. ∎

Next we give a criterion which is sufficient to guarantee that every compressing disk for ∂H has a trio of parallel arcs in C.

Proposition 3. *Suppose that for every pair of cutting disks D_i and D_j, $i \neq j$, and for each side of $\partial D_i \subseteq \partial H$, there is a collection of three parallel subarcs of C, with endpoints on the same side of ∂D_i, which on both ends cross D_j immediately before meeting D_i (see Figure 12). Then for every disk D in H, with $\partial D \subseteq \partial H$ transverse to C, (H,C) is not disk decomposable along D.*

Proof: Suppose that D is a decomposing disk for (H,C). By Proposition 2, D is not isotopic to any of the D_i. Because it is a compressing disk, ∂D cannot be trivial in ∂H. But since every simple loop in a pair of pants is either trivial or isotopic to one of the ∂-components, this means that ∂D cannot be isotoped to be disjoint from all of the ∂D_i; it would then lie in one of our two complementary pairs of pants.

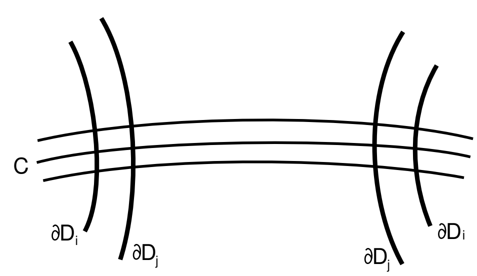

Figure 12

Consider an arc δ of $D \cap (D_1 \cup D_2 \cup D_3) \subseteq D$ which is outermost in D. The arc β of ∂D which δ cuts off then lies in one of our two pairs of pants, call it P. If β is a trivial arc in P, then, together with an arc α in one of the ∂D_k, it bounds a disk Δ in P. The suture C meets Δ in arcs, and, by applying the argument of the previous proposition, we may assume that each arc runs from α to β, since, if not, then either decomposing along D will create a trivial suture, implying that D is not a decomposing disk for H, or we can isotope ∂D across C without changing what the sutures in the sutured manifold obtained by splitting along D will look like. The pictures are identical to those of Figures 11a and b. A trivial arc cannot lie on the δ-side of Δ, by hypothesis.

But then, as before, we can isotope ∂D across Δ to reduce the number of points of intersection of ∂D with the ∂D_i, without changing the sutured manifold $(H|D, W)$. After repeatedly carrying out these isotopies, we can then assume that every outermost arc in D is non-trivial. By our argument above, there must be at least one non-trivial arc, α, since otherwise D, hence ∂D, is disjoint from the D_i.

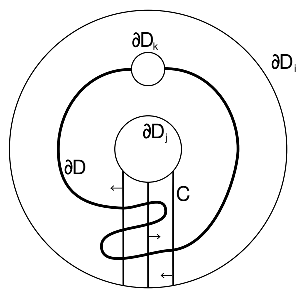

Figure 13

The arc β that α cuts off in ∂D, lying in one of the pairs of pants P, must therefore separate the two ∂-components ∂D_i and ∂D_j of P which it doesn't meet. It therefore must intersect the three arcs running from ∂D_i to itself, just before and after passing through ∂D_j, which were given by our hypothesis (Figure 13). As before, we may assume that β meets each arc of C running between ∂D_i and ∂D_j exactly once, since otherwise we can find a trivial subarc of β in $P|C$, allowing us, as before, to either reduce the number of points of intersection of β with C, or find a trivial suture after decomposing along D. But then by truncating the three arcs given by our hypothesis, by removing the short subarcs lying at the ends between ∂D_i and ∂D, we obtain three parallel arcs whose ends all lie on the same side of ∂D. Together with the obvious arcs in ∂D, they bound a rectangle R in ∂H.

These arcs in C may not lie in $\partial H|\partial D$ (Figure 14); but since we may, as above, assume that every other arc of $R \cap \partial D$ has no trivial intersections with our triple of arcs, some subrectangle bounded by arcs of ∂D will lie in $\partial H|\partial D$, with opposite transverse orientations on the ends. The intersection of this subrectangle with C will give us a triple of arcs with all of their ends on the same side of ∂D, giving us the triple of arcs in C which we need to apply Proposition 1. Therefore decomposing (H, C) along D will yield a trivial suture, so (H, C) is not disk decomposable along D.

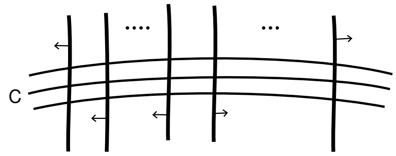

Figure 14

Note: We can weaken our hypotheses somewhat while still retaining the conclusion. From the proof we see the we need a trio of arcs which *either* end at D_i after passing through D_j *or* end at D_j after passing through D_i, since we really only need the fact that the ends of the arcs are passing *between* D_i and D_j. This gives us only half as many conditions to check.

§4
The examples

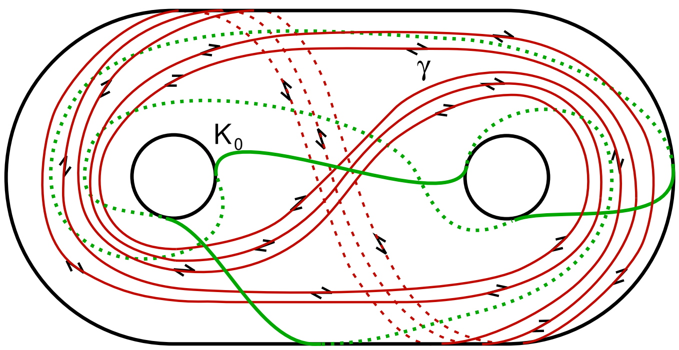

Figure 15

It is fairly easy to build examples of knots K satisfying the conditions of Propositions 1 and 3, by our initial Dehn twisting construction. We should note that the example given in Figure 7 does **not** satisfy the conditions of Proposition 3; there is no trio of arcs running from the middle disk which immediately run through the right hand disk on both ends. However, a still more complicated choice of initial twisting curve L will produce the examples we seek. Essentially, we need only make sure to choose a loop L so that, for every choice of a pair of cutting disks, there is **one** such arc in L; then the fact that Dehn twisting along L adds many parallel copies of L to K_0 will provide may parallel copies of each arc. One such example is given in Figure 15. It is easy to verify that for each choice of disk D_i, side of ∂D_i, and choice of disk D_j, $j \neq i$, there is an arc in L beginning and ending at D_i on the chosen side, which immediately passes through D_j at each end (or vice versa, which suffices for our purposes by the comment following the proof of Proposition 3). Properly chosen subarcs of the pair of arcs shown in Figures 16ab will suffice.

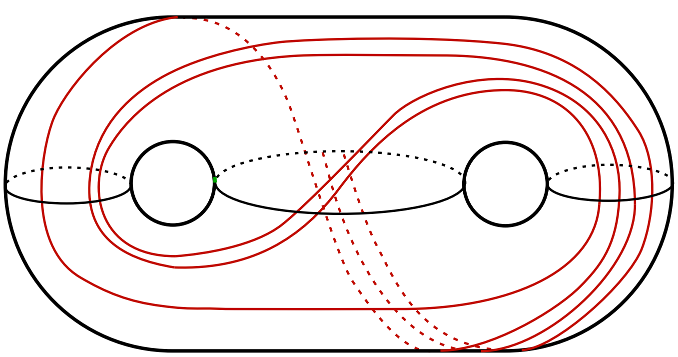

Figure 16a

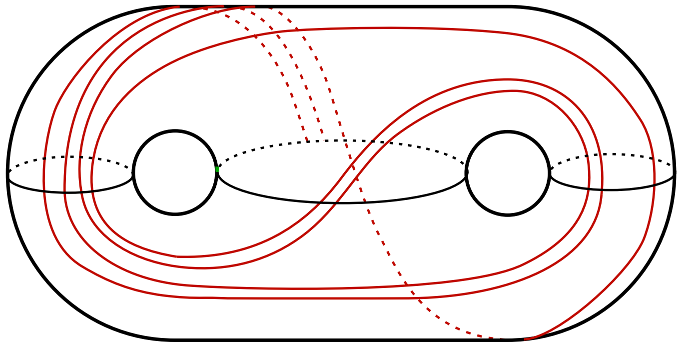

Figure 16b

To be certain that, when we perform a Dehn twist along L, the resulting loop C' will have at least three arcs parallel to each of the arcs given in the above figures, we must check that the *complement* of each arc α in L meets C at least three times. This is because as we traverse α, every time we cross C one of the arcs in C' parallel to α has been grafted to C and (we must assume) no longer runs parallel to α. Since we start with $|C \cap L|$ ($= 22$, in this case) arcs of C' running parallel to α at the start, and lose one at each crossing, we simply need to insure that we cross C no more than 19 times to ensure that three arcs will run parallel to α in C'. The reader can readily verify that for the arcs shown in Figure 16, the complementary arcs always meet C at least 6 times, by comparing with Figure 15.

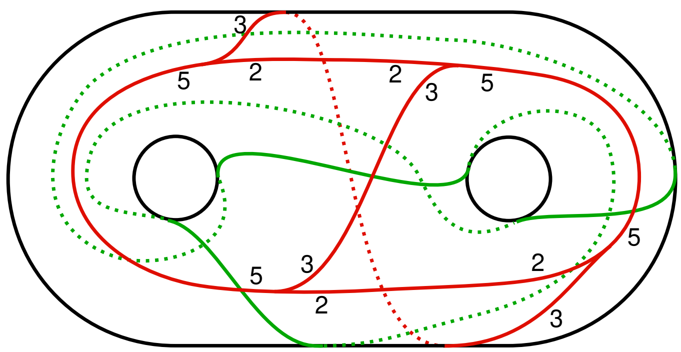

Figure 17a

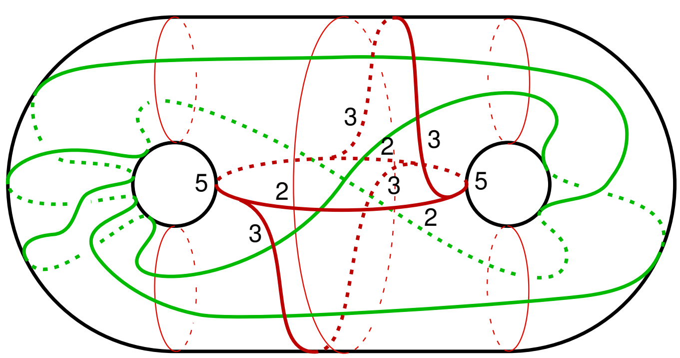

Figure 17b

To see what L looks like in the complement of our original Seifert surface Σ, we work with the train track τ in ∂H of Figure 8. L is carried by τ with weights 2, 3, and 5, as in Figure 17a. By keeping track of the intersections of τ with our cutting disks and K_0, we can reconstruct how τ would look in the interior version of our picture of X_{F_0}; see Figure 17b. This in turn allows us to reconstruct L, as it sits on our 4-punctured sphere P (Figure 18).

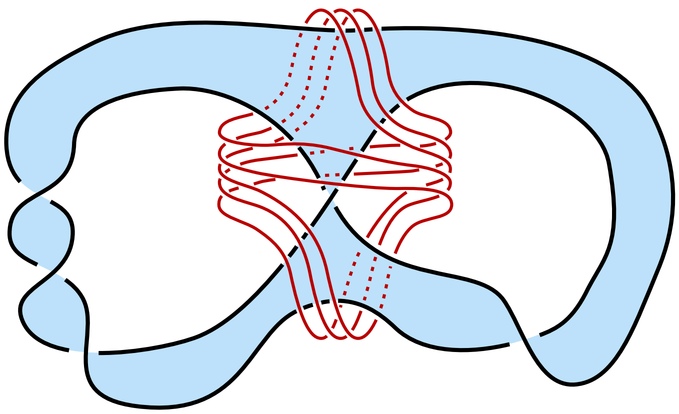

Figure 18

According to the computer program SnapPea [We], the knot K that we obtain from K_0 by 1/1 Dehn filling along L is hyperbolic; by the above work, the Seifert surface F_0 is carried under the Dehn filling to a genus one free Seifert surface F for K, which is not disk decomposable.

We can readily construct many more such examples, since any collection of *larger* weights on the train track of Figure 16b (which represent a connected loop, which essentially means that our replacements for 2 and 3 must be relatively prime) will also yield a knot and Seifert surface satisfying our theorem. Similarly, $1/n$ Dehn filling along L or these more complicated loops will also suffice, since more twists simply provide more parallel arcs for our arguments to use. We can also add full twists to the 'arms' of our original Seifert surface, without changing the essential features of the construction.

§4
Concluding remarks

The examples the we have obtained here, in some sense, manage to raise more questions than they answer. Perhaps the most pressing question raised is: do these knots that we build possess *other* Seifert surfaces which *are* disk decomposable? These other surfaces must, of course, also be free and have genus one. More generally, we might ask:

Question 1. *If the genus of K equals the free genus of K, does K always possess a disk decomposable Seifert surface?*

One way to show that the answer to this question is 'No' would be to show that some of our examples possess only one minimal genus (free) Seifert surface. There are several techniques for showing that a knot possesses a unique minimal genus Seifert surface (see, e.g., [Ko2],[Ko3],[KK]). Most of these can be phrased as saying that the knot K is 'simple enough'; since our approach to non-disk-decomposability is that the suture (= the knot) is 'complicated enough', applying such techniques will no doubt require some finesse.

While the Seifert surfaces that we build fail to be disk decomposable, they do have minimal genus, and so Gabai [Ga1] assures us that there is some sequence of decomposing surfaces which will split our sutured handlebody to trivially sutured 3-balls. What we have really shown here is that the first surface cannot be a disk. Since the decomposing surfaces must be incompressible, they will always (inductively) split our sutured handlebody at each stage to another sutured handlebody. The first splitting, then, *cannot reduce* the genus of the sutured handlebody (and, except for the case of a non-separating annulus, must *raise* it). An intersting question to ask, then , is: how high must the genus of the handlebody go? Are there, for example, (free) Seifert surfaces (of minimal genus) for which the <u>first</u> decomposing surface must raise the genus by an arbitrarily large amount?

Finally, we could attempt to strengthen our result by trying to replace 'disk decomposability' with something *weaker*. For example, a disk decomposable Seifert surface is always the leaf of a depth one foliation of the knot exterior [Ga2], and so the knot K must have depth [CC] (at most) one. So one can ask the *weaker* question:

Question 2. *If genus(K) = free genus(K), then does K have depth (at most) one?*

An answer of 'No' would be a stronger result. There is in fact a fairly simple necessary condition for a Seifert surface to be the leaf of a depth one foliation [CC]: the result of attaching a 2-handle to the suture of the associated sutured manifold must be the total space of a fiber bundle over the circle. See [Ko3] for an example which uses the Thurston norm to check this condition. Examples giving a negative answer to Question 1, which failed to satisfy this property, would also give a negative answer to Question 2.

Nowhere in our arguments is it really essential that our system of cutting disks consists of *three* disks. The exact same conditions used here, describing how the suture C meets a complete system of cutting disks for a higher genus handlebody, can therefore be used to find higher genus examples of sutured handlebodies which are not disk decomposable. One must use different arguments to show that the sutured handlebody is in fact taut; the conditions we impose only guarantee that the complement of the suture in ∂H is incompressible in H, and do not imply minimal genus.

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