Tautly foliated 3-manifolds with no R-covered foliations

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Abstract. We provide a family of examples of graph manifolds which admit taut foliations, but no **R**-covered foliations. We also show that, with very few exceptions, **R**-covered foliations are taut.

§0
Introduction

A foliation $\mathcal{F}$ of a closed 3-manifold M can be lifted to a foliation $\widetilde{\mathcal{F}}$ of the universal cover $\widetilde{M}$ of M; if the foliation $\mathcal{F}$ has no Reeb components, the leaves of this lifted foliation are all planes, and Palmeira [Pa] has shown that $\widetilde{M}$ is homeomorphic to R^3. Palmeira also showed that $\widetilde{\mathcal{F}}$ is homeomorphic to (a foliation of R^2 by lines)$\times R$. The space of leaves of $\widetilde{\mathcal{F}}$, the quotient space obtained by crushing each leaf to a point, is homeomorphic to the space of leaves of the foliation of R^2, and is a (typically non-Hausdoff) simply-connected 1-manifold.

If the space of leaves is Hausdorff (and therefore homeomorphic to **R**), we say that the foliation $\mathcal{F}$ is **R**-*covered.* Examples of **R**-covered foliations abound, starting with surface bundles over the circle; the foliation by fibers is **R**-covered. Thurston's notion of 'slitherings' [Th] also provide a large collection of examples. A great deal has been learned in recent years about **R**-covered foliations and the manifolds that support them (see, e.g., [Ba],[Ca],[Fe]), especially in the case when the underlying manifold M contains no incompressible tori.

The purpose of this note is to provide examples of 3-manifolds which admit taut foliations, but which do not admit any **R**-covered foliations. All of our examples are drawn from graph manifolds, and so all contain an incompressible torus, and, in fact, a separating one. The technology does not exist at present to identify examples which are atoroidal, and it is perhaps not clear that such examples should be expected to exist. That we can find the examples we seek among graph manifolds relies on two facts: (a) we have a good understanding [BR] of how taut and Reebless foliations can meet an incompressible torus, and (b) we understand [Br1],[Cl],[Na] which Seifert-fibered spaces can contain taut or Reebless foliations. These two facts have been used previously [BNR] to find graph manifolds which admit foliations with various properties, but no *"stronger"* properties; the examples we provide here are in fact the exact same examples used in [BNR] to illustrate its results. We will simply look at them from a somewhat different perspective.

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This paper can, in fact, be thought of as a further illustration of this ‘you can get this much, but no more’ point of view towards foliating manifolds. Namely, you can get taut, but not **R**-covered. **R**-covered is, it turns out, almost (but not quite) always a stronger condition: in all but a very small handful of instances, an **R**-covered foliation must be taut, as we show in the next section. This result seems to have been implicit in much of the literature on **R**-covered foliations; a different proof of this result, along somewhat different lines, can be found in [GS].

§1
R-COVERED ALMOST IMPLIES TAUT

In this section we show that in all but a very few instances, an **R**-covered foliation must be taut. We divide the proof into two parts; first we show that a Reebless **R**-covered foliation is taut, and then describe the manifolds that admit **R**-covered foliations with Reeb components.

Recall that a *Reeb component* is a solid torus whose interior is foliated by planes transverse to the core of the solid torus, each leaf limiting on the boundary torus, which is also a leaf. (There is a non-orientable version of a Reeb component, foliating a solid Klein bottle, which we will largely ignore in this discussion. It can be dealt with by taking a suitable double cover of our 3-manifold.) We follow standard practice and refer to both the solid torus and its foliation as a Reeb component. A foliation that has no Reeb components is called *Reebless*. A foliation is *taut* if for every leaf there is a loop transverse to the foliation which passes through that leaf. Taut foliations are Reebless.

Lemma 1. *If a closed, irreducible 3-manifold M admits a Reebless,* **R***-covered foliation $\mathcal{F}$ containing a compact leaf F, then every component of $M|F$, the manifold obtained by splitting M open along F, is an I-bundle over a compact surface. In particular, M is either a surface bundle over the circle with fiber F, or the union of two twisted I-bundles glued along their common boundary F.*

Proof: Because $\mathcal{F}$ is Reebless, the surface F is π_1-injective [No], and so lifts to a collection of planes in $\widetilde{M}$. Their image C in the space of leaves of $\widetilde{\mathcal{F}}$ is a discrete set of points in **R**, since it is closed, and any sequence of distinct points in C with a limit point can be used to find a sequence of points in F limiting on F in the transverse direction, contradicting the compactness of F. The complementary regions of F in M lift to the complementary regions of the lifts of F in $\widetilde{M}$; in the space of leaves they correspond to the intervals between successive points of C. Each is bounded by two points of C, and so every component $\widetilde{X}$ of the inverse image of a component X of $M|F$ has boundary equal to two lifts of F. Because $\widetilde{M}$ is simply-connected, as are the ∂-components of $\widetilde{X}$, $\widetilde{X}$ is simply-connected, and so $\widetilde{X}$ is the universal cover of X.

Because F π_1-injects into M, it π_1-injects into $M|F$, and hence into X. The index of $\pi_1(F)$ in $\pi_1(X)$ is equal to the number of connected components of the inverse image of F in the universal cover of X. To see this, choose a basepoint x_0 for X lying in F, and suppose that γ is a loop based at x_0 which is not in the image of $\pi_1(X)$. Then the lift $\widetilde{\gamma}$ of γ to $\widetilde{X}$ must have endpoints on distinct lifts of F, for otherwise the endpoints can be joined by an arc $\widetilde{\alpha}$ in the lift of F, whose projection, since $\widetilde{X}$ is simply connected (so $\widetilde{\gamma}*\widetilde{\alpha}$ is null-homotopic) is a null-homotopic loop $\gamma * \alpha$ in $\pi_1(X)$. This implies that $[\gamma] = [\bar{\alpha}] \in \pi_1(F)$, a contradiction. Choosing representatives from each coset of $\pi_1(F)$ in $\pi_1(X)$, and lifting each to arcs with initial points a fixed lift $\widetilde{x_0}$ of x_0, we find that their terminal points must therefore lie on distinct lifts of F.

But since $\widetilde{X}$ has only two boundary components, this means that $\pi_1(F)$ has index at most two in $\pi_1(X)$. Since X is irreducible (because F is incompressible and M is irreducible), [He, Theorem 10.6] implies that X is an I-bundle over a closed surface. The resulting description of M follows. $\square$

The foliation by fibers of a bundle over the circle is always taut. In the other case, when F separates, we understand [Br2] the structure of the foliation $\mathcal{F}$ on each of the two I-bundles, since their boundaries are leaves. If the induced foliations can be made transverse to the I-fibers of each bundle, then by taking a pair of I-fibers, one from each bundle, and deforming them so that they share endpoints on F, we can obtain a loop transverse to the leaves of $\mathcal{F}$, so $\mathcal{F}$ is taut. If the induced foliations cannot be made transverse to the I-fibers, then F is a torus, and (after possibly passing to a finite cover) the foliation contains a pair of parallel tori with a Reeb annulus in between. It is then straighforward to see that the resulting lifted foliation $\widetilde{\mathcal{F}}$ cannot be **R**-covered, since this torus$\times I$ will lift to $R^2 \times I$ whose induced foliation has space of leaves **R** *together with* two points (the two boundary components) that are *both* the limit of the positive (say) ray of the line. In particular, the space of leaves of $\widetilde{\mathcal{F}}$ would not be Hausdorff. Therefore:

Corollary 2. *A Reebless,* **R***-covered foliation is taut.* $\square$

We now turn our attention to **R**-covered foliations with Reeb components. Such foliations do exist, for example, the foliation of $S^2 \times S^1$ as a pair of Reeb components glued along their boundaries; the lift to $S^2 \times R$ consists of a pair of solid cylinders, each having space of leaves a closed half-line. Gluing the solid cylinders together results in gluing the two half-lines together, giving space of leaves **R**. We show, however, that, in some sense, this is the only such example. Recall that the Poincaré associate $P(M)$ of M consists of the connected sum of the non-simply-connected components of the prime decomposition of M, i.e., $M = P(M)\#$(a counterexample to the Poincaré Conjecture).

Lemma 3. *If $\mathcal{F}$ is an* **R***-covered foliation of the orientable 3-manifold M, which has a Reeb component, then $P(M) = S^2 \times S^1$.*

Proof: The core loop γ of the Reeb component must have infinite order in the fundamental group of M, otherwise the Reeb component lifts to a Reeb component of $\widetilde{\mathcal{F}}$; but since the interior of a Reeb component has space of leaves S^1, this would imply that S^1 embeds in **R**, a contradiction.

The Reeb solid torus therefore lifts to a family of infinite solid cylinders in $\widetilde{M}$, foliated by planes. The induced foliation of each closed solid cylinder has space of leaves a closed half-line properly embedded in the space of leaves of $\widetilde{\mathcal{F}}$. Each such half-line is disjoint from the others; but since **R** has only two ends, this implies that the Reeb component has at most two lifts to $\widetilde{M}$. This means that the inverse image of the core loop γ of the Reeb solid torus, in the universal cover $\widetilde{M}$, consists of at most two lines, and so the (infinite) cyclic group generated by γ has index at most 2 in $\pi_1(M)$. Because M is orientable, it's fundamental group is torsion-free, and so by [He, Theorem 10.7], $\pi_1(M)$ is free, hence isomorphic to $\mathbb{Z}$, and so [He, Exercise 5.3] $P(M)$ is an S^2-bundle over S^1. Since M is orientable, this gives the conclusion. $\square$

Note that the space of leaves in the universal cover does not change by passing to finite covers (there is only one universal cover), and so we can lose the orientability hypothesis by weakening the conclusion slightly. Putting the lemmas together, we get:

Corollary 4. *If a 3-manifold M admits an* $\mathbf{R}$*-covered foliation $\mathcal{F}$, then either $\mathcal{F}$ is taut or $P(M)$ is double-covered by $S^2 \times S^1$.* $\square$

§2
Taut but not $\mathbf{R}$-covered

In [BNR, Theorem D], the authors exhibit a family of 3-manifolds which admit $C^{(0)}$-foliations with no compact leaves, but no $C^{(2)}$-foliations without compact leaves. Each of the examples is obtained from two copies M_1,M_2 of (a once-punctured torus)$\times S^1$, glued together along their boundary tori by a homeomorphism A. What we will show is that for essentially the same choices of A, the resulting manifolds admit taut foliations, but no $\mathbf{R}$-covered ones.

A is determined by its induced isomorphism on first homology $\mathbb{Z}^2$ of the boundary torus, and so we will think of it as a 2×2 integer matrix with determinant ±1. We choose as basis for the homology of each torus the pair ($*\times S^1$,$\partial F\times *$), where F denotes the once-punctured torus. (Technically, we should orient these curves, but because all of the conditions we will encounter will be symmetric with respect to sign, the orientations will make no difference, and so we won't bother.)

Each M_i is a Seifert-fibered space (fibered by $*\times S^1$); the manifold M_A resulting from gluing via A is a Seifert-fibered space iff A glues fiber (1,0) to fiber (1,0), i.e., A is upper triangular. We will assume that this is not the case.

Let T denote the incompressible torus $\partial M_1 = \partial M_2$ in M_A. By [EHN], any horizontal foliation of M_i, i.e., a foliation everywhere transverse to the Seifert fibering of M_i, must meet ∂M_i in a foliation with slope in the interval (-1,1). [Note that this disagrees with the statement in [BNR], where the result was quoted incorrectly.]

If $\mathcal{F}$ is an $\mathbf{R}$-covered, hence Reebless, foliation of M, then by [BR], we can isotope $\mathcal{F}$ so that either it is transverse to T, and the restrictions $\mathcal{F}_i$ of $\mathcal{F}$ to M_i, $i = 1, 2$, have no Reeb or half-Reeb components, or $\mathcal{F}$ contains a cylindrical component, and therefore a compact (toral) leaf. In the second case, the torus leaf must hit the torus T, and is split into a collection of non-∂-parallel annuli; these (essential) annuli must be vertical in the Seifert-fibering of each M_i, since $F \times S^1$ contains no horizontal annuli. But this implies that the gluing map A glues fiber to fiber, a contradiction. Therefore, we may assume that $\mathcal{F}$ restricts to Reebless foliations on each of the manifolds M_i. Note that [BR] requires that we allow a finite amount of splitting along leaves to reach this conclusion; but since a splitting of an $\mathbf{R}$-covered foliation is still $\mathbf{R}$-covered (it amounts, in the space of leaves, to replacing points with closed intervals), this will not affect our argument.

By [Br3], each of the induced foliations $\mathcal{F}_i$ of M_i has either a vertical or horizontal sublamination. Every horizontal lamination in M_i can be extended to a foliation transverse to the fibers of M_i, and so meets ∂M_i in curves whose slope lies in $(-1, 1)$. If $\mathcal{F}_i$ has a vertical sublamination, it either meets ∂M_i in curves of slope ∞ (i.e., in curves homologous to (1,0)) or is disjoint from the boundary.

It is this last possibility, a vertical sublamination disjoint from T, that we wish to require, and so we will now impose conditions on the gluing map A to rule out the other possibilities. If *both* $\mathcal{F}_1$ and $\mathcal{F}_2$ have either a horizontal sublamination or a vertical sublamination meeting T, then for both M_1 and M_2, the induced foliations meet T in curves with slope in $(-1,1)\cup\{\infty\}$. Therefore, the gluing map

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{, which as a function on slopes, is } \mathrm{A}(x) = \frac{ax+b}{cx+d}$$

must have $A((-1,1)\cup\{\infty\})\cap((-1,1)\cup\{\infty\})\neq\emptyset$. But since

$$A(\infty)=\frac{a}{c},\ A(\frac{-d}{c})=\infty,\ A(\frac{1}{1})=\frac{a+b}{c+d}$$

and A is increasing if $ad-bc=1$ (take the derivative!), we can force $A((-1,1)\cup\{\infty\})$ to be disjoint from $(-1,1)\cup\{\infty\}$ by setting

$$|a|>|c|,\ |d|>|c|, \text{ and } \frac{a+b}{c+d}<-1$$

For example, we may choose $a=-n$, $b=-nm-1$, $c=1$, and $d=m$, with $n,m\geq 2$, so that

$$\frac{a+b}{c+d}=-n-\frac{1}{m+1}<-1$$

As the figure below shows, the conditions $-a>c$, $d>c$ and $ad-bc=1$ are also sufficient; what is needed, essentially, is that neither the graph of A nor either of its asymptotes pass through the square $[-1,1]\times[-1,1]$.

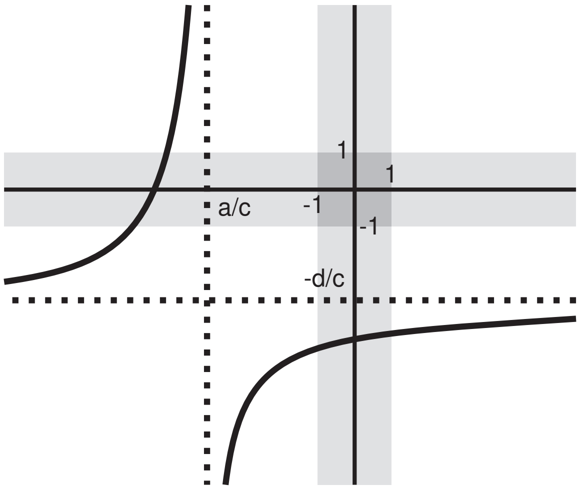

Figure 1

For such a gluing map A and resulting manifold M_A, either $\mathcal{F}_1$ or $\mathcal{F}_2$ (without loss of generality, $\mathcal{F}_1$) must contain a vertical sublamination $\mathcal{L}$ disjoint from $T=\partial M_1$. $\mathcal{L}$ is the saturation, by circle fibers, of a 1-dimensional lamination λ in the punctured torus F. This lamination λ cannot contain a closed loop, since then $\mathcal{F}_1$, and therefore $\mathcal{F}$, would contain a torus leaf L missing T. Lemma 1 would then imply that $M|L$ is an I-bundle, a contradiction, since it contains M_2. λ is therefore a lamination by lines. By Euler characteristic considerations, the complementary regions of λ, thought as in a torus, are products, and so the complentary region of λ in F which meets ∂F is topologically a (∂-parallel) annulus, with a pair of points removed from the 'inner' boundary. Therefore, the component N of $M_A|\mathcal{L}$ which contains T is homeomorphic to M_2 with a pair of parallel loops removed from ∂M_2 $=T$.

Now we will assume that $\mathcal{F}$ is $\mathbf{R}$-covered, and argue as in the proof of Lemma 1, to arrive at a contradiction. $\mathcal{L}$ lifts to a lamination in R^3 by planes, whose image in the space of leaves $\mathbf{R}$ of $\widetilde{\mathcal{F}}$ is a closed set. The two boundary leaves of $\mathcal{L}$ in ∂N are both annuli; they are the complements of the two parallel loops in ∂M_2 in the above description of $M_A|\mathcal{L}$. A lift of this complementary region to $\widetilde{M}$ has boundary consisting of lifts of the two annuli. Since the lift is a closed set in $\widetilde{M}$ its image in the space of leaves $\mathbf{R}$ is a connected, closed set, and therefore an interval. This implies that the lift of $M_A|\mathcal{L}$ has (at most) two boundary components, implying that the inverse image of each of the annulus leaves is a single (planar) leaf of $\widetilde{\mathcal{F}}$. This implies, as in the proof of Lemma 1, that the image in $\pi_1(M_A|\mathcal{L}) = \pi_1(M_2)$ of the fundamental group of each annulus has index at most 1, and so $\pi_1(M_2) = F_2 \times \mathbb{Z}$ is cyclic, a contradication. Therefore, $\mathcal{F}$ is not $\mathbf{R}$-covered. This implies:

Theorem 5. *With gluing map A given as above, M_A admits no $\mathbf{R}$-covered foliations.* $\square$

On the other hand, every manifold M_A built out of the pieces we have used admits taut foliations and, in fact, foliations with no compact leaves. We simply choose a vertical lamination with no compact leaves in each of the Seifert-fibered pieces M_i, missing the gluing torus T. The complement of this lamination is homeomorphic to $T \times I$, with a pair of parallel loops removed from each of the boundary components. Treating this as a sutured manifold, and thinking of this as $(S^1 \times I) \times S^1$, we can foliate it, transverse to the sutures, by parallel annuli. Then, as in [Ga1], we can spin the annular leaves near the sutures, to extend our vertical laminations to a foliation of M_A with no compact leaves. This give us:

Corollary 6. *There exist graph manifolds admitting taut foliations, but no $\mathbf{R}$-covered foliations.* $\square$

§3
$\mathbf{R}$-COVERED FINITE COVERS

Work of Luecke and Wu [LW] implies that (nearly) every connected graph manifold is *finitely covered* by a graph manifold that admits an $\mathbf{R}$-covered foliation. In particular, for any graph manifold M whose Seifert fibered pieces all have base surfaces having negative (orbifold) Euler characteristic, they find a finite cover M' (which is also a graph manifold) admitting a foliation $\mathcal{F}$ transverse to the circle fibers of each Seifert fibered piece of M', and which restricts on each piece to a fibration over the circle. Note that this implies that every leaf of $\mathcal{F}$ meets every torus which splits M' into Seifert-fibered pieces.

Even more, every leaf of the lift, to the universal cover of M, of $\mathcal{F}$ meets every lift P_1, P_2 of the splitting tori. This can be verified by induction on the number of lifts of the tori that we must pass through to get from a lift we know the leaf hits, to our chosen target lift. The initial step follows by picking a path $\widetilde{\gamma}$ between two 'adjacent' lifts P_1 and P_2, whose interior misses every lift of the splitting tori, and projecting down to M; this gives a path γ in a single Seifert fibered piece of M'. This path can be made piecewise vertical (in fibers) and horizontal (in leaves of $\mathcal{F}$), missing, without loss of generality, the multiple fibers of M' (just do this locally, in a foliation chart for $\mathcal{F}$; the Seifert fibering can be used as the vertical direction for the chart). Each vertical piece can then be dragged to the boundary tori, since the saturation by fibers of an edgemost horizontal piece of γ is a (singular) annulus with induced foliation by horizontal line segments; see Figure 2. [This is where the fact that $\mathcal{F}$ is everywhere transverse to the fibers is really used.] The end result of this process is a loop γ', homotopic rel endpoints to γ, which consists of two paths each lying in a circle fiber in the boundary tori, with a single path in a leaf of $\mathcal{F}$ lying in between. This lifts to a path homotopic rel endpoints to $\widetilde{\gamma}$, consisting of paths in the two lifted tori, and a path in some lifted leaf. This middle path demonstrates that *some* lifted leaf L hits both P_1 and P_2.

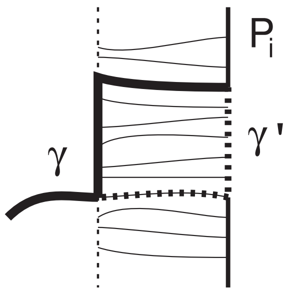

Figure 2

By choosing a point where any other lifted leaf L' hits a lift P of a splitting torus, and joining it by a path α to a point where L hits P_1 or P_2, we can apply the same straightening procedure as above (see Figure 3), to show that α is homotopic rel endpoints to paths, one of which lies in a lifted fiber and then lies totally in L', and the other of which lies totally in L, and then in a lifted fiber. This in particular implies that L' also hits P_1 and P_2. Therefore, every lifted leaf hits both P_1 and P_2, as desired.

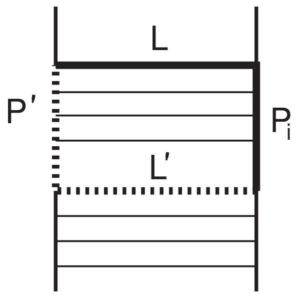

Figure 3

The inductive step is nearly identical; assuming our two leaves L_1 and L_2 both hit $P_1, \ldots, P_{n-1}$, and P_n can be reached from P_{n-1} without passing through any other lift of a splitting torus, the above argument implies that two leaves, in the lift of the relevant Seifert fibered piece, and contained in L_1 and L_2, hit both L_{n-1} and L_n, implying that L_1 and L_2 also both hit P_n.

But this in turn implies that the lifted foliation $\widetilde{\mathcal{F}}$ has space of leaves **R**. This is because the foliation induced by $\widetilde{\mathcal{F}}$ on any lift P of a splitting torus is a foliation transverse to (either of the) foliations by lifts of circle fibers, and so has space of leaves **R**, which can be identified with one of the lifts of a circle fiber. [This is probably most easily seen in stages: first pass to a cylindrical cover of the torus, for which the circle fibers lift homeomorpically. The induced foliation from $\mathcal{F}$ is by lines transverse to this fibering, and so has space of leaves one of the circle fibers. The universal cover P is a cyclic covering of this, whose induced foliation has space of leaves the universal cover of the circle fiber.] The argument above implies that every leaf of $\widetilde{\mathcal{F}}$ hits P at least once. But no leaf of $\widetilde{\mathcal{F}}$ can hit a lift of a circle fiber *more* than once; by standard arguments, using transverse orientability, a path in the leaf joining two such points could be used to build a (null-homotopic) loop transverse to $\widetilde{\mathcal{F}}$, contradicting tautness of $\mathcal{F}$, via Novikov's Theorem [No]. We therefore have a one-to-one correspondence between the leaves of $\widetilde{\mathcal{F}}$ and (any!) lift of a circle fiber in any of the Seifert-fibered pieces, giving our conclusion:

Proposition 7. *Any foliation of a graph manifold M, which restricts to a foliation transverse to the fibers of every Seifert-fibered piece of M, is* $\mathbf{R}$*-covered.* ☐

Combining this with the result of Luecke and Wu, we obtain:

Corollary 8. *Every graph manifold, whose Seifert-fibered pieces all have hyperbolic base orbifold, is finitely covered by a manifold admitting an* $\mathbf{R}$*-covered foliation.* ☐

Combining the proposition with our main resiult, we obtain:

Corollary 9. *There exist graph manifolds, admitting no* $\mathbf{R}$*-covered foliations, which are finitely covered by manifolds admitting* $\mathbf{R}$*-covered foliations.* ☐

§4
Concluding remarks

Being finitely covered by a manifold admitting an $\mathbf{R}$-covered foliation is nearly as good as having an $\mathbf{R}$-covered foliation yourself. Any property that could be verified in the presence of an $\mathbf{R}$-covered foliation, which remains 'virtually' true (e.g., virtually Haken, or having residually finite fundamental group), would then be true of the original manifold. It would then be of interest to know:

Question 1. *Does every 3-manifold with universal cover R^3 have a finite cover admitting an* $\mathbf{R}$*-covered foliation?*

Or, even stronger:

Question 2. *Does every irreducible 3-manifold with infinite fundamental group have a finite cover admitting an* $\mathbf{R}$*-covered foliation?*

Weaker, but still interesting:

Question 3. *Does every tautly foliated 3-manifold have a finite cover admitting an* $\mathbf{R}$*-covered foliation?*

The first two questions could be broken down into Question 3 and

Question 4. *Does every 3-manifold (in the appropriate class) have a tautly foliated finite cover?*

Note that showing that every irreducible 3-manifold with infinite fundamental group has a tautly foliated finite cover would settle the

Conjecture. *Every irreducible 3-manifold with infinite fundamental group has universal cover R^3.*

Questions 1 and 2 can be thought of as weaker versions of the (still unanswered) question, due to Thurston, of whether or not every hyperbolic 3-manifold is finitely covered by a bundle over the circle; the foliation by bundle fibers is **R**-covered. Gabai [Ga2] has noted that there are Seifert-fibered spaces for which the answer to Thurston's question is 'No', although an observation of Luecke and Wu [LW] implies that, via the results [EHN], the answer to our Question 1 is 'Yes', for Seifert-fibered spaces, since [Br4] a transverse foliation of a Seifert-fibered space is **R**-covered. [Note that the arguments of the previous section can be modified to give a different proof of this; look at how lifted leaves meet lifts of a single regular fiber, instead of lifts of the splitting tori.] Question 4, with its conclusion replaced by 'have a taut foliation', has as answer 'No'; examples were first found among Seifert-fibered spaces [Br1],[Cl]; there are no known examples among hyperbolic manifolds.

Question 5. *Do there exist hyperbolic 3-manifolds admitting no taut foliations?*

Finally, the result we have established here for graph manifolds is still unknown for hyperbolic 3-manifolds:

Question 6. *Do there exist hyperbolic 3-manifolds which admit taut foliations, but no* **R***-covered foliations?*

References

[Ba] T. Barbot, *Caractérisation des flots d'Anosov en dimension 3 par leurs feuilletages faibles.*, Ergodic Theory Dynam. Systems **15** (1995), 247-270.

[BNR] M. Brittenham, R. Naimi, and R. Roberts, *Graph manifolds and taut foliations*, J. Diff. Geom. **45** (1997), 446-470.

[BR] M. Brittenham and R. Roberts, *When incompressible tori meet essential laminations*, Pacific J. Math **190** (1999), 21-40.

[Br1] M. Brittenham, *Essential laminations in Seifert-fibered spaces*, Topology **32** (1993), 61-85.

[Br2] ———, *Essential laminations in* $\mathbb{I}$*-bundles*, Transactions of the AMS **349** (1997), 1463-1485.

[Br3] ———, *Essential laminations in Seifert-fibered spaces: boundary behavior*, Topology Appl. **95** (1999), 47-62.

[Br4] ———, *Exceptional Seifert-fibered spaces and Dehn surgery on* 2*-bridge knots*, Topology **37** (1998), 665-672.

[Ca] D. Calegari, *Foliations and the geometry of 3-manifolds*, Thesis, University of California at Berkeley (2000).

[Cl] W. Claus, *Essential laminations in closed Seifert-fibered spaces*, Thesis, Univ. of Texas at Austin (1991).

[EHN] D. Eisenbud, U. Hirsch, and W. Neumann, *Transverse Foliations on Seifert Bundles and Self-homeomorphisms of the Circle*, Comment. Math. Helv. **56** (1981), 638-660.

[Fe] S. Fenley, *Anosov flows in 3-manifolds*, Annals of Math **139** (1994), 79-115.

[Ga1] D. Gabai, *Foliations and the topology of 3-manifolds*, J. Diff. Geom. **18** (1983), 445-503.

[Ga2] ———, *On* 3*-manifolds finitely covered by surface bundles*, Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984), Cambridge Univ. Press, 1986, pp. 145-155.

[GS] S. Goodman and S. Shields, *A condition for the stability of* **R***-covered on foliations of 3-manifolds*, Trans. AMS **352** (2000), 4051-4065.

[He] J. Hempel, *3-Manifolds*, Annals of Math. Studies No. 86, Princeton University Press, 1976.

[Na] R. Naimi, *Foliations transverse to fibers of Seifert manifolds*, Comment. Math. Helv. **69** (1994), 155-162.

[No] S. Novikov, *Topology of foliations*, Trans Moscow Math Soc **14** (1965), 268-305.

[Pa] C. Palmeira, *Open manifolds foliated by planes*, Annals of Math **107**, 109-121.
[Th] W. Thurston, *Three-manifolds, Foliations and Circles, I*, preprint.
[LW] J. Luecke and Y.-Q. Wu, *Relative Euler number and finite covers of graph manifolds*, Geometric topology (Athens, GA, 1993), Amer. Math. Soc., 1997, pp. 80-103.

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