Surfaces in low-dimensional topology

Proposed Projects

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Surfaces have played a significant role in the development of 3-manifold topology since the very beginnings of the subject. Perhaps the first significant result along these lines was Kneser's proof [Kn] that every closed 3-manifold M can be decomposed along finitely many 2-spheres into irreducible pieces. This was also the first proof which made use of normal forms for surfaces with respect to a triangulation of M.

Injective surfaces rose to prominence in 3-manifold topology in the 1960's with work of Waldhausen, who proved nearly every major conjecture about the structure of 3-manifolds, in the case that the 3-manifold M (is irreducible and) contains an injective surface F. If M is irreducible and has infinite fundamental group, then it has long been conjectured that the universal cover of M must be R^3, and any irreducible 3-manifold homotopy equivalent to M must also be homeomorphic to M. Waldhausen [Wa] proved both of these conjectures, if one adds the hypothesis that M contains an injective surface; we call such a manifold *Haken*. These results have lent great weight to the validity of these conjectures in general. In the same vein, much of the evidence in support of Thurston's Geometrization Conjecture rests largely on Thurston's proof [Th1] of this conjecture for Haken manifolds.

Taut foliations provide another effective tool for the study of 3-manifolds. Work of Gabai [Ga1] shows that every surface in M which has minimal genus in its homology class is the leaf of a taut foliation of M; the converse, that every compact leaf of a taut foliation has minimal genus, is due to Thurston [Th2]. These results proved to be the main tools needed to show that satellite knots have Property P [Ga2], and to solve the Property R Conjecture for knots [Ga3], which was proved by building a taut foliation in the manifold obtained by 0-frame surgery on a knot. The concepts of sutured manifold decomposition, and thin position, which were first introduced in this work, have had a deep and far-reaching influence on the development of low-dimensional topology, especially in knot theory.

Essential laminations provide a simultaneous generalization of injective surfaces and taut foliations, and have proved to have similar power in unlocking the structure of the 3-manifolds which contain them. Gabai and Oertel [GO] showed that any 3-manifold containing an essential lamination has universal cover R^3; Palmeira [Pa] had previously shown this for tautly foliated 3-manifolds. Gabai and Kazez [GK] have shown that atoroidal manifolds containing genuine essential laminations have Gromov negatively curved fundamental group, giving a 'weak' form of Thurston's Geometrization Conjecture for these manifolds.

The principal investigator's research has largely dealt with essential laminations, focusing on how to construct them, what they can tell us about the topology of the 3-manifold containing them, and how the structure of the 3-manifold containing one is manifested in the structure of the lamination. We plan to continue to show how laminations and lamination-theoretic ideas can help us to understand the topology of 3-manifolds, especially in areas related to the Virtual Haken Conjecture and exceptional Dehn surgeries on knots and 3-manifolds. We also plan to continue to explore the structure of sutured manifold decompositions, and the associated notion of the depth of a knot. These projects are outlined in more detail below.

After discussing, in section 0, some of the basic concepts related to the proposed research, in section 1 we describe a program for showing that many manifolds with essential laminations are virtually Haken, and how this implies both geometrization and Waldhausen's theorem for these manifolds. In section 2 we discuss a program to use essential laminations to study exceptional Dehn surgeries, and Seifert-fibered surgeries in particular. In section 3 we discuss sutured manifold hierarchies, and a project for building knots with large depth. In section 4 we discuss the broader impact of the proposed research. In section 5 we describe research carried out under previous NSF support.

0. Notation and definitions

For ease of exposition in the project descriptions in later sections, we collect here some of the more common definitions and concepts, involving the objects we will consider.

$\mathcal{L}$ will always stand for a lamination, $\mathcal{F}$ is a codimension-one foliation, B is a branched surface, K is a knot in the 3-sphere S^3, F is a compact surface, M is a compact 3-manifold, γ is a simple closed curve, and I is the unit interval [0,1]. The exterior of an object J, that is, the complement of an open neighborhood of J, will be denoted $X(J)$.

A properly embedded surface F in M is *incompressible* if its inclusion induces an injection on the level of fundamental groups, and it is not a 2-sphere. F is *∂-incompressible* if inclusion induces an injection on the level of relative fundamental groups. F is *injective* if it satisfies both conditions. A *Seifert surface* for a knot K is an orientable surface with boundary equal to K. A Seifert surface which has minimal genus among all Seifert surfaces for K is incompressible in $X(K)$. A manifold is *reducible* if it contains a 2-sphere which does not bound a 3-ball; otherwise it is *irreducible*. A manifold is *toroidal* if it contains an embedded incompressible torus; otherwise it is *atoroidal*. A manifold is *hyperbolic* if it admits a complete metric of constant sectional curvature -1. A *Seifert-fibered space* is a compact 3-manifold which can be foliated by circles. A Seifert-fibered space is *exceptional* if it does not contain an embedded incompressible torus. A *graph manifold* is a manifold obtained by gluing Seifert-fibered spaces together along their boundary tori.

A codimension-one *foliation* $\mathcal{F}$ of a 3-manifold M is a way of expressing M as a disjoint union of (usually non-compact) surfaces called *leaves*, which locally run parallel to one another. A *lamination* $\mathcal{L}$ is a codimension-one foliation of a closed subset of M. A lamination is *genuine* if it has at least one complementary region which is not an I-bundle over a (usually non-compact) surface, with associated ∂I-bundle consisting of leaves of $\mathcal{L}$. A foliation $\mathcal{F}$ is *Reebless* if no leaf is a compressible torus; a foliation is *taut* if through every leaf there is a loop γ which is everywhere transverse to the leaves of $\mathcal{F}$. Taut foliations are Reebless. In both cases, the leaves of $\mathcal{F}$ inject into M on the level of π_1.

A *branched surface* is a finite two-dimensional complex B with a well-defined tangent plane at each point. The non-manifold points of B are a union of closed loops intersecting transversely; these loops are called the *branch curves* of B. The points of intersection of the branch curves are called the *triple points* of B. The boundary of $N(B)$ splits naturally into the *horizontal* boundary, consisting of the boundary points of each I-fiber, and the *vertical* boundary, consisting of annuli. The annuli correspond to the branch curves of B.

The *sectors* of B consists of the metric completions of the connected components of $B\setminus$(the branch curves). They are compact surfaces with corners at the triple points of B.

A lamination is said to be *carried* by a branched surface B if it can be embedded in an I-fibered neighborhood $N(B)$ of B so that each leaf is everywhere transverse to the I-fibers. $\mathcal{L}$ is *carried with full support* if it meets every I-fiber of $N(B)$. A lamination $\mathcal{L}$ is *essential* if it is carried with full support by an *essential branched surface*; we refer the reader to [GO] for a definition of this object. The most important property that an essential lamination shares with injective surfaces and taut foliations is that each of its leaves inject on the level of fundamental groups. A 3-manifold which contains an essential lamination is called *laminar*.

Dehn surgery along a knot K consists of removing a neighborhood of the knot, leaving a manifold with boundary a torus, and then gluing a solid torus $D^2 \times S^1$ to the boundary torus. The surgery is determined by which simple closed curve γ on the boundary torus is glued to the meridian disk of the solid torus. γ is called the *slope* of the surgery. *Dehn filling* is the same procedure, except that the neighborhood of the knot has already been removed.

A *sutured manifold* is a 3-manifold M together with a collection of disjoint simple loops γ in ∂M, which split ∂M into surfaces R_+ and R_-, with $\partial M = R_+ \cup R_-$ and $R_+ \cap R_- = \gamma$. The *sutures* γ consist of cores of the vertical boundary annuli. M can be split open, or *decomposed*, along any surface F transverse to γ [Ga1], to obtain a new sutured manifold. We call F a *decomposing surface* for (M, γ).

A surface F in a 3-manifold M is in *normal form* with respect to a triangulation or cell decomposition of M if it meets each 3-cell Δ of M in disks, so that each disk meets each 1-cell in $\partial\Delta$ at most once. A lamination $\mathcal{L}$ is in normal form if each of its leaves is in normal form.

1. Laminar manifolds are virtually Haken

Nearly all of the most important outstanding conjectures in 3-manifold topology have been proved for Haken manifolds [Wa],[Th1]. But most irreducible 3-manifolds, in some sense, are not Haken [Ha1]. It has long been conjectured, however, that every irreducible 3-manifold M with infinite fundamental group has a Haken covering space with finite degree; such an M is said to be *virtually Haken*. An even stronger conjecture asserts that M has a finite cover $\widetilde{M}$ with infinite first homology; this is equivalent to the assertion that the fundamental group of $\widetilde{M}$ surjects onto the integers Z. In this case we say that $\pi_1(M)$ (and, by extension, M) is *virtually Z-representable*. Several recent papers [BZ1],[CL],[DT],[Ma] ,[MMZ] have demonstrated the existence of such finite covers for several classes of manifolds, using a variety of techniques, from immersed injective surfaces to Heegard surfaces to the properties of $PSL_2(\mathbf{C})$.

If a manifold M with no $Z \times Z$ in its fundamental group is virtually Haken, then Thurston's geometrization theorem implies that M has a hyperbolic finite cover, and so, by [Ga4],[Ga5] M is itself hyperbolic. In addition, any irreducible 3-manifold homotopy equivalent to M is homeomorphic to M [Ga5]. Consequently, finding a Haken finite cover for M implies that M is geometrizable and satisfies Waldhausen's Theorem, at least in the (algebraically) atoroidal case. (The same is (almost) true for the toroidal case, as well.) Therefore, showing that a laminar manifold M is virtually Haken would have a very large payoff; it implies the geometrizability of M, and that any irreducible manifold homotopy equivalent to M is homeomorphic to M.

We propose to prove the virtual Haken conjecture, for a broad class of laminar 3-manifolds. A lamination $\mathcal{L}$ is *simple* if it is carried by a branched surface B having no triple points. Previous work of the proposer [Br1],[Br2] provides many examples of such manifolds; in fact, non-trivial Dehn surgery on nearly half of the knots in the standard tables [Rl] will yield a manifold with a simple essential lamination. The laminations constructed in these examples are in fact *very simple*, that is, the branched surfaces B have, in addition, only a single branch curve. In the end, it will be the combinatorics of these branched surfaces B, and not the fact that they carry laminations, which we will exploit to prove the conjecture. The outline below therefore focuses on the branched surfaces, as the main object of study.

Problem: *Prove the virtual Haken conjecture, for the class of 3-manifolds containing a simple essential branched surface.*

For the purposes of the conjecture, we may assume that M is not itself Haken (since otherwise we are done). Then by [Br3] we know that the exterior $X(B)$ of B in M consists of handlebodies. The fundamental group of M is therefore a quotient of $\pi_1(B)$, obtained by adding relators corresponding to a system of compressing disks for the handlebodies. For a simple branched surface, $\pi_1(B)$ is very straightforward to describe. Our main approach will be to identify finite covers of M by finding representations of $\pi_1(B)$ to finite groups, for which the added relators are sent to the identity. In the initial stages of the project, we plan to focus on very simple branched surfaces, since these already provide a very large class of 3-manifolds to which the results will apply.

First, we will need to identify the possible sets of relators which will arise from the branch curves of an essential, very simple, branched surface. In our setting, the property that B is essential boils down to the fact that the complement of the branch curves, in the boundary of the handlebodies $X(B)$, is incompressible in $X(B)$. Two essentially different methods have been developed to characterize the incompressibility of $\partial H \setminus \gamma$ in a handlebody H. The first is largely geometric, using the notion of *waves* [St]. The second is algebraic, and uses the notion of a group element (corresponding to γ) *binding* the fundamental group of H [Ly]. Each of these, in the end, involve the existence of a set of compressing disks meeting γ in an appropriate way. These properties should help us to get a handle on the kinds of relators that will arise in our constructions.

Problem: *Determine the structure of $\pi_1(M)$ as a quotient of $\pi_1(B)$ for B a simple essential branched surface. Use this to build finite covers of M, by finding representations of $\pi_1(B)$ to finite groups.*

Showing that a 3-manifold M has a non-trivial finite cover is a highly non-trivial task; it is, in fact, still an open problem in general. Our approach will be to search for covers in which the lift of the branch curve is a single curve; the total space of such a cover will then still satisfy the same hypothesis, and so we can recursively build an infinite chain of finite covers. We will begin by looking at specific classes of Dehn surgeries on knots; many of the knot surgeries successfully treated in [MMZ], for example, are known to contain simple laminations [Br2]. By examining the structure of the covers, from the point of view of the simple branched surface B, we expect to begin to develop the kind of insight necessary to build finite covers, in more generality.

Once we have developed techniques to construct finite covers, we will need to be able to establish that they are Haken. From the point of view of virtual Z-representability, this amounts to showing that a map of the fundamental group to Z is onto. Again, this should be possible to verify, by having a sufficiently good understanding of the structure of the relators needed, for π_1 of the lift of B, to present the fundamental group of the cover.

Problem: *Develop methods to establish the existence of essential surfaces in manifolds containing simple essential branched surfaces.*

We will also pursue a more geometric approach to this problem, by using the lift $\widetilde{B}$ of the branched surface B to build essential surfaces in the cover. We cannot expect $\widetilde{B}$ itself to carry a compact surface; if we interpret the existence of the surface as a system weights on the sectors of $\widetilde{B}$, then by projecting down we obtain a system of weights on B, and hence a compact surface. But most simple branched surfaces carry no compact surfaces. Instead we must look for surfaces *transverse* to $\widetilde{B}$. M can be viewed as being built from the handlebody complements H of B, via gluings along the incompressible surfaces $\partial H \setminus \gamma$. Such a structure is inherited by the covers of M. This structure has many characteristics in common with both bundles over the circle and Heegaard splittings; in both cases, there is much that can be said about incompressible surfaces in M [CG],[CJR],[FH],[He],[Hh]. Corresponding results are likely to hold in our context.

2. Exceptional Dehn surgeries on knots

A great deal of recent research in 3-manifold topology owes its motivation to Thurston's Hyperbolic Dehn Surgery Theorem [Th3], which states that all but finitely many Dehn fillings along the boundary torus of a hyperbolic 3-manifold are hyperbolic. This has led to a great deal of research into when, and in particular, *how often* the Dehn filling on a hyperbolic manifold or knot fails to be hyperbolic. Thurston's Geometrization Conjecture [Th1] asserts that failure to be hyperbolic can be detected topologically; a non-hyperbolic closed 3-manifold must either have finite π_1, contain an essential sphere or essential torus, or be an exceptional Seifert-fibered space.

There is now a very good picture describing when most of these possibilities can occur; these results are largely described in terms of how *far apart* (measured as the minimal intersection number of the two Dehn filling curves) exceptional surgeries of each of the four types can be. For the first three types, the papers [BZ2],[Go],[Wu1],[Oh],[GL],[CGLS] taken together provide a nearly complete picture of how such exceptional surgeries occur.

To date however, the remaining case, exceptional Seifert fibered spaces, has proved to be far more difficult. We showed [Br4], using essential laminations, that at most 20 Dehn fillings can be finite, reducible, toroidal, or exceptional Seifert fibered, but this has since been eclipsed [Ag],[La] in the general case by results using techniques based on the $2\pi-$Theorem [BH] and negatively-curved groups.

Our approach in [Br4] was to use the observation [Br5] that exceptional Seifert fibered spaces do not contain *genuine* essential laminations. Using a construction of essential laminations in hyperbolic 3-manifolds [GM],[Mo], we observed that all Dehn fillings off of four lines in Dehn surgery space contain genuine essential laminations. Combined with the

$2\pi-$Theorem, this gave our bounds on exceptional surgeries. This approach was later used in [BGZ] to understand Seifert-fibered surgery slopes of Haken hyperbolic 3-manifolds.

We can therefore use essential laminations to determine that Dehn surgery on a knot does not yield an exceptional Seifert-fibered space, by building genuine essential laminations in the surgered manifolds [Br5]. Wu [Wu2] has shown that one can similarly use essential laminations to spot Dehn fillings that have no injective tori. In joint work with Wu [BW], we used these approaches to completely classify manifolds obtained by Dehn surgery on 2-bridge knots, according to whether they are reducible, toroidal, exceptional Seifert-fibered, have finite fundamental group, or are hyperbolic.

There are only a few other classes of knots whose Dehn surgeries are completely characterized in this way. Torus knot surgeries can be understood directly [Ms], and surgeries on some families of arborescent knots can be classified [Wu3] using Thurston's Geometrization Theorem (since the surgeries are all Haken). We propose to use essential laminations, and in particular the techniques of [Br5] and [Wu2], to carry out similar classifications, using the approach developed in [BW]. In particular, we propose to apply these techniques to the laminations built by Delman and Roberts [DR] for alternating knot surgeries, to produce a classification of the manifolds obtained by Dehn surgery on alternating knots.

Problem: *Classify the Dehn surgeries on alternating knots, according to their geometric structures.*

In some cases this will involve developing new families of essential laminations, to apply these techniques directly. There are two basic methods for constructing essential laminations in knot surgeries. With the first, one builds infinitely many essential laminations in the exterior $X(K)$ of K, meeting the boundary in loops, which 'cap off' to give essential laminations in each Dehn filling along K. For example, this is the approach taken by Hatcher [Ha2], for 2-bridge knots, and Roberts [Ro], for most alternating knots. The other approach is to build a single essential lamination $\mathcal{L}$ in the complement of K, which remains essential under every non-trivial Dehn surgery along K, known as a *persistent* lamination. This is the approach taken by Delman [De] for two-bridge knots, by Wu [Wu3] for arborescent knots, and by the proposer [Br1],[Br2] for several infinite families of knots. The latter method seems to be more suited to the classification of exceptional surgeries, in light of [Br5] and [Wu2]; the results of those papers are based largely on an understanding of the structure of the component of the complement of $\mathcal{L}$ containing the knot K. We therefore expect that an efficient approach to our problem will involve building persistent laminations for the alternating knots for which the only known examples of laminations come from Roberts' constructions. We expect that the techniques of [Br1],[Br2], building persistent laminations for knots from incompressible Seifert surfaces of "simpler" knots, will succeed in most cases, since incompressible Seifert surfaces for alternating knots can be built by Seifert's algorithm [Se].

Problem: *Construct persistent laminations for all non-torus alternating knots.*

3. Hyperbolic knots with high depth

Given a knot K in S^3 and a Seifert surface Σ for K, Gabai [Ga1] showed that if Σ has least genus among all Seifert surfaces for K, then Σ is the sole compact leaf of a finite depth foliation of the exterior $X(K)$ of K. Thurston [Th2] showed that the converse is true: a compact leaf of a taut foliation has minimal genus in its homology class. A foliation $\mathcal{F}$ has finite depth if all leaves are proper (i.e., no leaf limits on itself in the transverse direction) and there is an upper bound to a chain of proper inclusions $\overline{L_0} \subseteq \overline{L_1} \subseteq \cdots \subseteq \overline{L_n}$ of closures of leaves of $\mathcal{F}$. n is called the *depth* of the foliation. The smallest n among all foliations having Σ as sole compact leaf is called the *depth of* Σ, and the smallest depth among all minimal genus Seifert surfaces for K is called the *depth of* K.

The practical value of Gabai's and Thurston's results are that they show that a Seifert surface can be certified to have minimal genus by building a finite depth foliation for which it is a leaf, and that such a certification is always possible. Gabai used this technology, for example, to compute the genera of the knots through 10 crossings [Ga6], to give a geometric proof that Seifert's algorithm applied to a reduced alternating projection of a knot yields a least genus Seifert surface [Ga7], and to compute the genera of the arborescent knots [Ga8].

Gabai's result, showing that a finite depth foliation can be built around a minimal genus surface, proceeds by building the foliation "one leaf at a time" (each leaf raising the depth by one). With appropriate choices, the complements of the leaves added up to each point can be shown to be "more nearly a product (surface)$\times I$" than at the previous stage, in a precise sense; when the complement <u>is</u> a product, the remainder can be foliated without raising the depth any further. The sequence of surfaces chosen in building the foliation of $X(K)$ constitutes a sutured manifold hierarchy [Ga1] of the knot exterior $X(K)$. The depth of Σ (resp. K), which corresponds to the length of the sutured manifold hierarchy, can therefore be interpreted as the fewest number of added leaves needed to produce a product complement, and so measures how far $X(\Sigma)$ (resp. $X(K)$) is from a product (resp., a bundle over the circle).

In all of Gabai's examples above, the depth of the foliation built by Gabai is at most one. Thus all of these knots have depth at most one; they are at most "one step away" from being fibered. A natural question, then, is whether or not there are knots with depth greater than one. This was answered in the affirmative by Cantwell and Conlon [CC1], who showed, in fact, that there are knots of arbitrarily large depth. The essential point is that the untwisted double K' of a non-trivial knot K has depth at least one higher than the depth of K. Iterated doubling therefore gives knots of arbitrarily large depth. These knots, however, have many essential tori in their complements; this is, in essence, why the depth is so high. Cantwell and Conlon therefore asked if there is a knot with atoroidal complement, that is, a hyperbolic knot, with depth greater than one.

In [Ko], Kobayashi gave an example of such a knot K; in particular, he constructed a hyperbolic knot K with a unique minimal genus Seifert surface Σ which is the leaf of a depth 2, but not of a depth 1, foliation of $X(K)$. His proofs rely on a combinatorial argument to show that the surface is unique, and a computation of the unit ball in the Thurston norm of $X(K)$, along with a result of Cantwell and Conlon [CC2], to show that Σ is not the leaf of a depth 1 foliation. Finally, he explicitly builds a depth 2 foliation for which Σ is a leaf.

In recent work [Br6], we have shown how to reproduce much of Kobayashi's results, in a much broader context. We start with a construction of knots K with free genus-1

Seifert surfaces [Br7], that is, the exterior $X(\Sigma)$ of Σ is a genus-2 handlebody. The knot K, thought of as lying in the boundary of $X(\Sigma)$, determines a word w in the free group on two letters $F(a,b) = \pi_1(X(\Sigma))$. We demonstrate that the uniqueness of the Seifert surface can be inferred from the combinatorics of the word w, using the fact that non-uniqueness implies the existence of distinct disjoint Seifert surfaces [ST]. Then we apply an algorithm of K. Brown [Bro] for computing the Bieri-Neumann-Strebel invariant [BNS] of the one-relator group $\langle a, b : w\rangle$ to develop conditions on w sufficient to imply that the surface Σ is not the leaf of a depth one foliation, utilizing the same result of Cantwell and Conlon above. Infinitely many (in some sense, in fact, most) of the knots from the construction pass through these two "filters", to emerge as examples of knots with depth at least two. All of the resulting knots are hyperbolic. For both "filters", combinatorial properties of the word w ensure that the property that we desire is satisfied.

It is not known if any or all of these surfaces built in this way have depth two. In fact, we strongly suspect that most of them have depth greater than two, and so the resulting knots have depth greater than two. To date, however, there is no technique to show that a Seifert surface, and hence a knot, has depth greater than two. We plan to develop such techniques, in the context in which our examples have been found, namely, among free minimal genus Seifert surfaces.

> **Problem:** *Develop conditions under which, for $n > 2$, a free Seifert surface cannot be the leaf of a depth n foliation.*

In this context, all of the topology of the situation is captured by the word w representing the knot K in $\pi_1(X(\Sigma))$, and so the property of having depth greater than two is, in theory, a group-theoretic one; we propose to develop sufficient conditions on w for the depth of K to be at least n, using combinatorial group theory. As a starting point, there are higher geometric invariants of groups [BiR], similar to the BNS invariant, which may shed light on this. Initially we will search for isolated examples, as Kobayashi did. But the goal is to find broad families of knots with high depth, if they exist, as we managed to do for depth greater than one. An exploration of some of the simpler examples arising from the construction above should help to provide insight into the kinds of sutured manifold hierarchies we should focus on. Recent work of Agol and Li [AL] provides an algorithmic approach to the construction of such hierarchies.

The main goal of this project is to understand the process of sutured manifold decomposition; Gabai's construction of a foliation "one leaf at a time" is, after all, really the construction of a sutured manifold decomposition of the knot exterior. We therefore expect that our studies of the depth of knots will shed light on the process of sutured manifold decomposition as a whole, and specifically to provide insight into when such a decomposition must have large length. There is an analogous concept of a hierarchy for a Haken 3-manifold, which puts fewer restrictions on the surfaces used in the decomposition; in that case it is known [Ja] that every Haken manifold has a hierarchy of length at most four. We do not expect such a result in our context, although any result along similar lines would be both surprising and potentially useful. Recent work [HKM] has demonstrated how sutured manifold decompositions can give rise to tight contact structures, and so this project has the potential to shed light on their behavior, as well.

4. Broader impact and integration of research and education

The principal investigator carries out many activities designed to communicate mathematical ideas to a wide audience. He regularly presents his work to research-level audiences at colloquia, meetings, and conferences. The investigator has also given talks to both undergraduate and high school audiences; the topics of these talks are informed by, and in many instances a direct result of, his research endeavors.

The investigator has also organized conferences for both specialized and general mathematical audiences. He has co-organized two special sessions at AMS meetings. He has a recurring role at the lead organizer for the annual Regional Workshops in the Mathematical Sciences at the University of Nebraska. These workshops have the goal of informing students about research in the mathematical sciences, especially among undergraduates considering graduate work in the mathematical sciences. They also provide a forum for contact between University of Nebraska students and faculty and students and researchers in mathematics, statistics, and computer science at undergraduate institutions within a region encompassing Nebraska, Kansas, Missouri, Iowa, North and South Dakota, and beyond.

The investigator maintains a website linking together resources within the low-dimensional topology community, which is visited approximately 250 times each week. The most-often visited portions of the site include a comprehensive list of low-dimensional topologists' homepages, and links to upcoming meetings and conferences.

The investigator expects that parts of the last project described in this proposal will prove to be suitable for the involvement of undergraduates in a directed research project. In particular, the combinatorial aspect of the words w in the fundamental group of the handlebody, and their group-theoretic significance, are suitable for such explorations. The University of Nebraska has several programs in place to provide funding for undergraduates engaged in research with faculty; the investigator plans to apply for such funding at the appropriate stage of the project.

5. Results from Prior NSF support

During 1997 - 2000 the proposer was supported by NSF grant number DMS-9704811, originating at Vassar College, which was transferred to the University of North Texas in the second year as NSF grant number DMS-9896215. During the period of the grant he carried out research which has resulted in seven papers. He also had an additional seven papers appear in or be accepted for publication in refereed journals; all of these papers underwent some revision during the grant period.

During the contract period we finished a collection of new constructions of laminations in knot complements, which remain essential under every non-trivial Dehn filling along the knot; these laminations are called *persistent* for the knot. These constructions were motivated by an example of Oertel [Oe] of a persistent lamination in the complement of the knot 5_2 (in Rolfsen's tables [Rl]). In the paper *Persistently laminar tangles* [Br1] we showed how to associate Oertel's lamination with a certain tangle T, in the sense that this lamination would also be persistent for any knot obtained by tangle sum with T. We also showed how to build similar laminations for several families of related tangles. This paper has appeared in J. Knot Thy. Ram.

Oertel's lamination also turned out to be the first of an infinite family of laminations, in an altogether different sense. We showed, in the paper *Persistent laminations from Seifert surfaces* [Br2], how to use an incompressible Seifert surface F for a knot K to build a lamination that is persistent for an infinite family of knots built from K and F. Oertel's lamination, it turns out, is the lamination that would be built by this construction from a disk spanning the unknot. We then instituted a search, aided by the computer program SnapPea [We], of the knots in the standard knot tables that could be persistently laminated by this construction. To date this has succeeded in finding persistent laminations for slightly less than half of the knots in the standard tables. This paper has appeared in J. Knot Thy. Ram.

The work of the previous paragraph required an incompressible Seifert surface for a knot as input, and this motivated us to study the most well-known method for generating Seifert surfaces: Seifert's algorithm [Se]. We were interested in discovering when Seifert's algorithm would build an *incompressible* Seifert surface for a knot, and in particular whether or not it could build *every* incompressible (free) Seifert surface for a knot. We were able to determine the answer to this second question - no - by finding a relationship between the genus of a surface built by Seifert's algorithm and the volume of the knot complement. In particular, we showed that if Seifert's algorithm can build a surface of genus g for the hyperbolic knot K, then the complement of K can have volume at most $122g$. This implies that if a knot has large volume, then Seifert's algorithm, applied to any projection of K, must always build a surface of high genus. This result was written up in the paper *Bounding canonical genus bounds volume* [Br8].

On the other hand, we found a method for generating hyperbolic knots which have incompressible free Seifert surfaces of genus one, but arbitrarily large volume. The method involved repeatedly doing Dehn filling on certain unknotted loops in the complement of an incompressible free genus one surface F_0 for an initial knot K_0, e.g., the 'planar' surfaces for a pretzel knot. The filling carries F_0 to a new incompressible free Seifert surface for a new knot K; we then used Thurston's Geometrization Theorem [Th1] and estimates of Adams [Ad] to ensure that the knot K is hyperbolic and has large volume. Together with the previous paper, this provides examples of knots with incompressible free Seifert surfaces which cannot be obtained by Seifert's algorithm for any projection of K. These results were written in the paper *Free genus one knots with large volume* [Br7], which has appeared in Pacific J. Math..

Another source of incompressible free Seifert surfaces comes from the disk decomposable surfaces of Gabai [Ga7]; the process of disk decomposition was introduced as a practical method for computing the genus of a knot K. If a surface is disk decomposable, it is automatically free and has minimal genus among all Seifert surfaces for K. We then asked whether the converse to this is true: must a free Seifert surface for K which has minimal genus be disk decomposable? Using work of Goda [Gd], who gave a sufficient condition for a sutured handlebody to *not* be disk decomposable, we were able to show that several families of Seifert surfaces built along the lines of the previous paper were free, had minimal genus for their boundaries, and were not disk decomposable. These results were written up in the paper *Free Seifert surfaces and disk decompositions* [Br9], which has appeared in Math. Zeit.

We also carried out collaborative work with a group of mathematicians in Japan, begun at a workshop on laminations held at Nara Women's University and organized by Tsuyoshi Kobayashi. We were interested in exploring the interplay between normal forms for laminations and persistent laminations for knots. We showed how to apply the existence of essential laminations in normal form with respect to a regular cell decomposition for a knot [Br10] to build persistent laminations for a knot, using a very standard cell decomposition arising from a projection of the knot, with two 3-cells consisting essentially of the 3-balls lying above and below the projection plane. The structure of the normal disks turn out to have intriguing connections with the work of Menasco and Thistlethwaite [MT] on incompressible surfaces in the complement of an alternating knot. This work was written up in the paper *Essential laminations and branched surfaces in the exteriors of links* [BHHKS], which has been submitted for publication.

We also explored the connection between taut foliations and R-covered foliations. In the paper *Tautly foliated 3-manifolds with no R-covered foliations* [Br11], we showed that certain graph manifolds have taut foliations but no R-covered ones. On the other hand, they all have *finite covers* which admit R-covered foliations. This last observation follows from a result of Luecke and Wu [LW]; they show that most graph manifolds admit finite covers having foliations which restrict on each Seifert-fibered piece to foliations transverse to the circle fibers. We show that any such foliation is R-covered, extending our previous result about foliations transverse to the fibers of a Seifert-fibered space [Br5]. This paper has appeared in the Proceedings of the Conference on Foliations: Geometry and Dynamics, Warsaw, 2000.

In addition to these projects, seven papers appeared in print or were accepted for publication during the period of the grant, which underwent varying degrees of revision during the grant period. These papers were largely the result of two projects: studying the structure of essential laminations in Seifert-fibered spaces and in 3-manifolds containing injective tori, and studying how essential laminations in the complement of a knot K provide information about the structure of the manifolds obtained by Dehn surgery on K.

The paper *Graph manifolds and taut foliations* [BNR], joint with Ramin Naimi and Rachel Roberts, explored the extent to which the gluing maps between boundary components of Seifert-fibered spaces, used to build a graph manifold M, affect the topological and smoothness properties of the taut and Reebless foliations of M. We discovered that the gluing maps could restrict the behavior of the manifold's foliations in very unexpected ways.

The results of this paper were supported by three others. In *When incompressible tori meet essential laminations* [BR], joint with Rachel Roberts, we showed that a taut or Reebless foliation could almost always be split along an incompressible torus to give taut or Reebless foliations in each piece of the split open manifold, with the exception of one well-known and well-understood case. In *Essential laminations in Seifert-fibered spaces : Boundary behavior* [Br12] and *Essential laminations in* I*-bundles* [Br13], we completed work begun in [Br14] to describe the structure of essential laminations in Seifert-fibered spaces. Together these papers showed that essential laminations in Seifert-fibered spaces are, with a few well-known exceptions, always everywhere transverse to the circle fibers of the fibering. Together with work of Jankins and Neumann [JN] and Naimi [Na], these papers provide a determination of all of the boundary slopes of essential laminations in Seifert-fibered spaces.

Taken together, the second paper shows that a taut foliation (for example) in a graph manifold must come from taut foliations in its Seifert-fibered pieces; the third and fourth papers show how those foliations must meet the boundary tori of the pieces, i.e, what range of boundary slopes would be possible. The first paper then analyzed how the gluing maps could match up these collections of boundary slopes, to determine when a taut foliation could exist in the first place.

The other main project dealt with using essential laminations in the complement of a knot to determine the geometric structure of the manifolds obtained by Dehn surgery on the knot. Gabai and Mosher [Mo] showed that every cusped hyperbolic 3-manifold M contains an essential lamination $\mathcal{L}$ which remains essential under 'most' Dehn fillings of the cusp. In the first of our papers, *Essential laminations, exceptional Seifert-fibered spaces, and Dehn filling* [Br5], we showed how these laminations could be used to improve, to 20, the previous bound of 24 [BH] on the number of Dehn fillings of M which could be reducible, toroidal, finite-π_1, or exceptional Seifert-fibered. This bound has since been improved upon by later work of other [Ag],[La], but the techniques of this paper can still provide better bounds, when more information about the lamination $\mathcal{L}$ is known.

We also used similar techniques to study surgeries on 2-bridge knots. In the paper *Exceptional Seifert-fibered spaces and Dehn surgery on 2-bridge knots* [Br5] we showed that non-integral surgery on a non-torus 2-bridge knot could never yield an exceptional Seifert fibered space, confirming, for these knots, a conjecture of Cameron Gordon. In the paper *The classification of Dehn surgeries on 2-bridge knots* [BW], joint with Ying-Qing Wu, we extended this work to completely classify the surgeries on 2-bridge knots, determining when they are reducible, toroidal, exceptional Seifert-fibered, or hyperbolic.

Bibliography

[Ad] C. Adams, *Volumes of N-cusped hyperbolic 3-manifolds* J. London Math. Soc. **38** (1988) 555–565.

[Ag] I. Agol, *Bounds on exceptional Dehn filling*, Geom. Topol. **4** (2000) 431–449.

[AL] I. Agol and T. Li, *An algorithm to detect laminar 3-manifolds*, preprint.

[BGZ] S. Boyer, C. Gordon and X. Zhang, *Dehn fillings of large hyperbolic 3-manifolds*, J. Differential Geom. **58** (2001) 263–308.

[BH] S. Bleiler and C. Hodgson, *Spherical space forms and Dehn filling*, Topology **35** (1996) 809–833.

[BHHKS] M. Brittenham, C. Hayashi, M. Hirasawa, T. Kobayashi, and K. Shimokawa, *Essential laminations and branched surfaces in the exteriors of links*, preprint.

[BNR] M. Brittenham, R. Naimi, and R. Roberts, *Graph manifolds and taut foliations*, J. Diff. Geom. **45** (1997) 446–470.

[BNS] R. Bieri, W. Neumann and R. Strebel, *A geometric invariant of discrete groups*, Invent. Math **90** (1987) .451–477

[BiR] R. Bieri, and B. Renz, *Valuations on free resolutions and higher geometric invariants of groups*, Comment. Math. Helv. **63** (1988) 464–497.

[Br1] M. Brittenham, *Persistently laminar tangles*, J. Knot Theory Ram. **8** (1999) 415–428.

[Br2] M. Brittenham, *Persistent laminations from Seifert surfaces*, J. Knot Thy. Ram. **10** (2001) 1155-1168.

[Br3] M. Brittenham, *Essential laminations in non-Haken* 3*-manifolds*, Topology Appl. **53** (1993) 317–324.

[Br4] M. Brittenham, *Essential laminations, exceptional Seifert-fibered spaces, and Dehn filling*, J. Knot Thy. Ram. **7** (1998) 425–432.

[Br5] M. Brittenham, *Exceptional Seifert-fibered spaces and Dehn surgery on* 2*-bridge knots.* Topology **37** (1998) 665–672.

[Br6] M. Brittenham, *Knots with unique minimal genus Seifert surface and depth of knots*, preprint.

[Br7] M. Brittenham, *Free genus one knots with large volume*, Pacific J. Math. **201** (2001) 61–82.

[Br8] M. Brittenham, *Bounding canonical genus bounds volume*, preprint.

[Br9] M. Brittenham, *Free Seifert surfaces and disk decompositions*, Math. Zeit. **240** (2002) 197–210.

[Br10] M. Brittenham, *Essential laminations and Haken normal form : Regular cell decompositions*, preprint.

[Br11] M. Brittenham, *Tautly foliated 3-manifolds with no R-covered foliations*, Proceedings of the Conference on Foliations: Geometry and Dynamics, Warsaw, 2000.

[Br12] M. Brittenham, *Essential laminations in Seifert-fibered spaces : Boundary behavior*, Topology Appl. **95** (1999) 47–62.

[Br13] M. Brittenham, *Essential laminations in* I*-bundles*, Trans. Amer. Math. Soc. **349** (1997) 1463–1485.

[Br14] M. Brittenham, *Essential laminations in Seifert-fibered spaces*, Topology **32** (1993) 61–85.

[BR] M. Brittenham and R. Roberts, *When incompressible tori meet essential laminations*, Pacific J. Math. **190** (1999) 21–40.

[Bro] K. Brown, *Trees, valuations and the Bieri-Neumann-Strebel invariant*, Inv. Math. **90** (1987) 479–504.

[BW] M. Brittenham and Y.-Q.- Wu, *The classification of Dehn surgeries on 2-bridge knots*, Comm. Anal. Geom. **9** (2001) 97–113.

[BZ1] S. Boyer and X. Zhang, *Virtual Haken* 3*-manifolds and Dehn filling*, Topology **39** (2000) 103–114.

[BZ2] S. Boyer and X. Zhang, *On Culler-Shalen seminorms and Dehn filling*, Annals of Math. **148** (1998) 737–801.

[CC1] J. Cantwell and L. Conlon, *Depth of knots*, Topology Appl. **42** (1991) 277–289.

[CC2] J. Cantwell and L. Conlon, *Foliations of* $E(5_2)$ *and related knot complements*, Proc. AMS **118** (1993) 953–962.

[CG] A. Casson and C. Gordon, *Reducing Heegard splittings*, Topology Appl. **27** (1987) 275–283.

[CGLS] M. Culler, C. Gordon, J. Luecke and P. Shalen, *Dehn surgery on knots*, Annals of Math. **125** (1987) 237–300.

[CJR] M. Culler, W. Jaco and H. Rubinstein, *Incompressible surfaces in once-punctured torus bundles*, Proc. London Math. Soc. **45** (1982) 385–419.

[CL] D. Cooper and D. Long, *Virtually Haken Dehn-filling* J. Diff. Geom. **52** (1999) 173–187.

[De] C. Delman, *Essential laminations and Dehn surgery on* 2*-bridge knots*, Topology Appl. **63** (1995) 201–221.

[DR] C. Delman and R. Roberts, *Alternating knots satisfy Strong Property P*, Comm. Math. Helv. **74** (1999) 376–397.

[DT] N. Dunfield and W. Thurston, *The Virtual Haken Conjecture: Experiments and Examples*, preprint.

[FH] W. Floyd and A. Hatcher, *Incompressible surfaces in punctured-torus bundles*, Topology Appl. **13** (1982) 263–282.

[Ga1] D. Gabai, *Foliations and the topology of 3-manifolds*, J. Diff. Geom. **18** (1983) 445–503.

[Ga2] D. Gabai, *Surgery on knots in solid tori*, Topology **28** (1989) 1–6.

[Ga3] D. Gabai, *Foliations and the topology of 3-manifolds, III*, J. Diff. Geom. **26** (1987) 479–536.

[Ga4] D. Gabai, *Homotopy hyperbolic* 3*-manifolds are virtually hyperbolic*, J. Amer. Math. Soc. **7** (1994) 193–198.

[Ga5] D. Gabai, *On the geometric and topological rigidity of hyperbolic* 3*-manifolds*, J. Amer. Math. Soc.**10** (1997) 37–74.

[Ga6] D. Gabai, *Foliations and genera of links*, Thesis, Princeton University, 1980.

[Ga7] D. Gabai, *Foliations and genera of links*, Topology **23** (1984) 381–394.

[Ga8] D. Gabai, *Genera of the arborescent links*, Mem. Amer. Math. Soc. **59** (1986) 1–98.

[Gd] H. Goda, *A construction of taut sutured handlebodies which are not disk decomposable*, Kobe J. Math **11** (1994) 107–116.

[GK] D. Gabai and W. Kazez, *Group negative curvature for* 3*-manifolds with genuine laminations*, Geom. Topol. **2** (1998) 65–77.

[GL] C. Gordon and J. Luecke, *Reducible manifolds and Dehn surgery*, Topology **35** (1996) 385–409.

[GM] D. Gabai and L. Mosher, presonal communication.

[Go] C. Gordon, *Boundary slopes of punctured tori in 3-manifolds*, Trans. Amer. Math. Soc. **350** (1998) 1713–1790.

[GO] D. Gabai and U. Oertel, *Essential laminations in 3-manifolds*, Annals of Math. **130** (1989), 41–73.

[Ha1] A. Hatcher, *On the boundary curves of incompressible surfaces*, Pacific J. Math. **99** (1982) 373–377.

[Ha2] A. Hatcher, *Some examples of essential laminations in* 3*-manifolds*, Ann. Inst. Fourier (Grenoble) **42** (1992) 313–325.

[He] J. Hempel, *Coverings of Dehn fillings of surface bundles*, Topology Appl. **24** (1986) 157–170.

[Hh] D. Heath, *On Haken's algorithm and immersed surfaces*, Kobe J. Math. **13** (1996) 1–7.

[HKM] K. Honda, W. Kazez and G. Matić, *Tight contact structures and taut foliations*, Geom. Topol. **4** (2000) 219–242.

[Ja] W. Jaco, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Mathematics **43**, American Mathematical Society, Providence, R.I., 1980.

[JN] M. Jankins and W. Neumann, *Rotation numbers of products of circle homeomorphisms*, Math. Ann. **271** (1985) 381–400.

[Ko] T. Kobayashi, *Example of hyperbolic knot which do not admit depth 1 foliation*, Kobe J, Math **13** (1996) 209–221.

[Kn] H. Kneser, *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*, Jahresber. Deutch. Math. Verien. **38** (1929) 248–260.

[La] M. Lackenby, *Word hyperbolic Dehn surgery*, Invent. Math. **140** (2000) 243-282.

[LW] J. Luecke and Y-Q. Wu, *Relative Euler number and finite covers of graph manifolds*, Geometric topology (Athens, GA, 1993) 80–103.

[Ly] Lyon, *Incompressible surface in the boundary of a handlebody - an algorithm*, Canad. J. Math **32** (1980) 590–595.

[Ma] J. Masters, *Virtual homology of surgered torus bundles*, Pac. J. Math. **195** (2000) 205–223.

[MMZ] J. Masters, W. Menasco and X. Zhang, *Heegaard splittings and virtually Haken Dehn filling*, preprint.

[Mo] L. Mosher, *Laminations and flows transverse to finite depth foliations*, preprint.

[Ms] L. Moser, *Elementary surgery along a torus knot*, Pacific J. Math. **38** (1971) 737–745.

[MT] W. Menasco and M. Thistlethwaite, *Surfaces with boundary in alternating knot exteriors*, J. Reine Ang. Math. **426** (1992) 47–65.

[Na] R. Naimi, *Foliations transverse to fibers of Seifert manifolds*, Comment. Math. Helv. **69** (1994) 155–162.

[Oe] U. Oertel, *Affine laminations and their stretch factors*, Pacific J. Math. **182** (1998) 303–328.

[Oh] S. Oh, *Reducible and toroidal manifolds obtained by Dehn filling*, Topology Appl. **75** (1997) 93–104.

[Pa] C. Palmeira, *Open manifolds foliated by planes*, Annals of Math. **107** (1978), 109–121.

[Rl] D. Rolfsen, *Knots and Links*, Publish or Perish Press, 1976.

[Ro] R. Roberts, *Constructing taut foliations*, Comment. Math. Helv. **70** (1995) 516–545.

[Se] H. Seifert, *Über das Geschlecht von Knoten*, Math. Annalen **110** (1934) 571–592.

[ST] M. Scharlemann and A. Thompson, *Finding disjoint Seifert surfaces*, Bull. LMS **20** (1988) 61–64.

[Th1] W. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. (1982) 357–381.

[Th2] W. Thurston, *A norm for the homology of* 3*-manifolds*, Mem. Amer. Math. Soc. **59** (1986), i–vi and 99–130.

[Th3] W. Thurston, *The Geometry and Topology of 3-manifolds*, Princeton University lecture notes (1978).

[Wa] F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Annals of Math. **87** (1968) 56–88.

[We] J. Weeks, *SnapPea, a program for creating and studying hyperbolic 3-manifolds*, available for download from http://www.northnet.org/weeks/ .

[Wu1] Y.-Q. Wu, *Dehn fillings producing reducible manifolds and toroidal manifolds*, Topology **37** (1998) 95–108.

[Wu2] Y.-Q. Wu, *Sutured manifold hierarchies, essential laminations, and Dehn surgery*, J. Diff. Geom. **48** (1998) 407–437.

[Wu3] Y.-Q. Wu, *Dehn surgery on arborescent knots*, J. Diff. Geom. **43** (1996) 171–197.